Exact Solution to a Class of Generalized Kitaev Spin-1/2 Models in Arbitrary Dimensions

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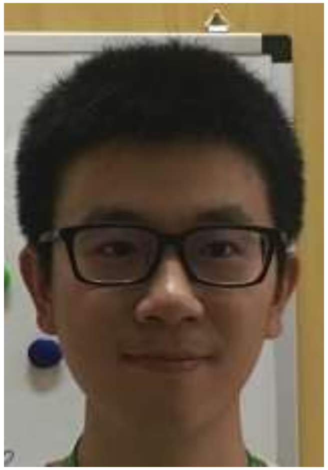
Collaborators



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References:

[1] J. J. Miao, H. K. Jin, F. Wang, F. C. Zhang, **YZ**, arXiv:1806.06495 (2018).

[2] J. J. Miao, H. K. Jin, F. C. Zhang, **YZ**, arXiv: 1806.10960 (2018).

Outline

- A brief introduction to Kitaev honeycomb model
- The construction of exactly solvable models
- Generating new models: 1D, 2D and 3D
- A particular example in 2D: a Mott insulator model
- 3D examples and possible realization in real materials

Kitaev Honeycomb model

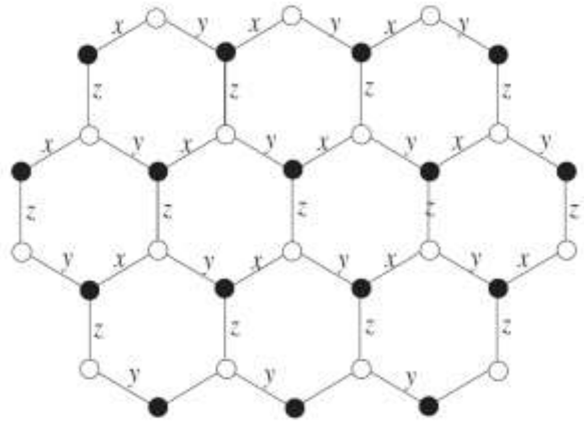
Spin-1/2 model (compass model)

$$H = -J_x \sum_{x\text{ link}} K_{ij} - J_y \sum_{y\text{ link}} K_{ij} - J_z \sum_{z\text{ link}} K_{ij}$$

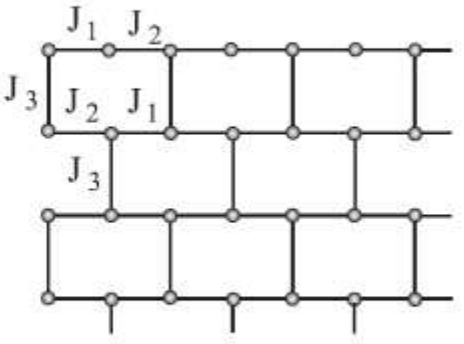
where K_{ij} is defined as

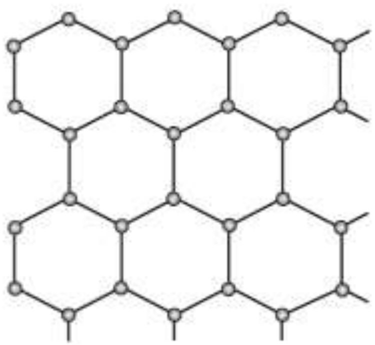
$$K_{ij} = \begin{cases} \sigma_i^x \sigma_j^x, & \text{if } (i,j) \text{ is a } x \text{ link}, \\ \sigma_i^y \sigma_j^y, & \text{if } (i,j) \text{ is a } y \text{ link}, \\ \sigma_i^z \sigma_j^z, & \text{if } (i,j) \text{ is a } z \text{ link}. \end{cases}$$

Kitaev (2006)



Brick-wall representation





Feng, Zhang, Xiang (2007); Chen, Nussinov (2008)

- Exact solvability
- Quantum paramagnet
    - *SU(2)* invariant ground state
    - Emergent *SU(2)* symmetry
- Fractional spin excitations
- Topologically distinct phases
- Two spins per unit cell

Existing generalizations

- ✓ **Spin-1/2 models in 2D**
  - ✓ Yao, Kivelson (2007); Yang, Zhou, Sun (2007); Baskaran, Santhosh, Shankar (2009); Tikhonov, Feigelman (2010); Kells, Kailasvuori, Slingerland, Vala (2011); …
- ✓ **Spin-1/2 models in 3D**
  - ✓ Si, Yu (2007); Ryu (2009); Mandal, Surendran (2009); Kimchi, Analytis, Vishwanath (2014); Nasu, Udagawa, Motome (2014); Hermanns, O'Brien, Trebst (2015); Hermanns, Trebst (2016); …
- ✓ **Multiple-spin interactions**
  - ✓ Kitaev (2006); Lee, Zhang, Xiang (2007); Yu, Wang (2008); …
- ✓ ***SU(2)*-invariant models**
  - ✓ F. Wang(2010); Yao, Lee (2011); Lai and O. I. Motrunich (2011); …
- ✓ **Higher spin models**
  - ✓ Yao, Zhang, Kivelson (2009); Wu, Arovas, Hung (2009); Chern (2010); Chua, Yao, Fiete (2011); Nakai, Ryu, Furusaki (2012); Nussinov, van den Brink, (2013); …

Our goals

- Provide some generic rules for searching generalized Kitaev spin-1/2 models in arbitrary dimensions.
- Constrict ourselves on spin-1/2 models.
- Demonstrate some models of particular interest.

Construction of spin-1/2 models

Basic idea: ① Construct exactly solvable 1D spin chains and ② then couple them to form a connected lattice in arbitrary dimensions.

Steps:

① Construct spin-1/2 chains that can be exactly solved by the Jordan-Wigner transformation.

② Couple these chains to form a connected lattice on which the spin-1/2 model can be still exactly solved by the Jordan-Wigner transformation.

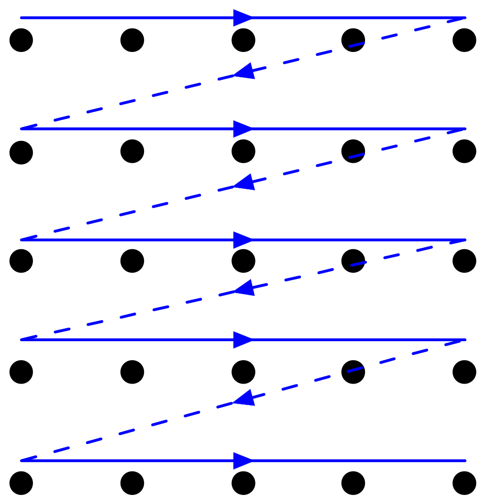
Parquet rules:

① Elementary rules

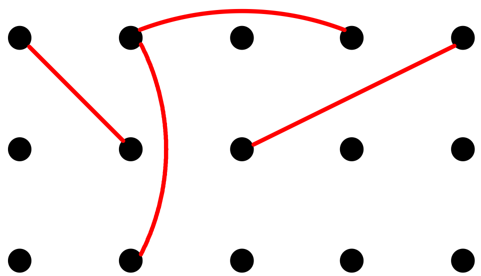
② Supplementary rules

Sites and links on a lattice

- **Consider a *d*-dimensional cube**, $d = 2, 3, 4, \cdots$
- **Site labelling**: $n = (n_1, n_2, \cdots, n_d),\ 1 \le n_j \le L_j,\ j = 1,\cdots,d$
- **Ordering of sites**
  - Define a number, $N = n_1 + \sum_{j=2}^{d}(n_j - 1)\prod_{l=1}^{j-1} L_j$, for each site n;
  - If $N < M$, then $n < m$.
- **Link**: a pair of sites (n, m)
  - Local link: $\sum_{j=1}^{d}|n_j - m_j| = 1$
  - Nonlocal link: $\sum_{j=1}^{d}|n_j - m_j| > 1$



ordering of sites



local and nonlocal links

Construction rules

Model Hamiltonian

$$H = H_{local}^{(2)} + H_{nonlocal}^{(2)} + H_{nonlocal}^{(M)}$$

Interactions

- $H_{local}^{(2)}$: local two−spin terms, $J_{n,n+\hat{1}}^{\alpha\beta}\sigma_n^{\alpha}\sigma_{n+\hat{1}}^{\beta}$ and $J_{n,n+\hat{k}}^{zz}\sigma_n^{z}\sigma_{n+\hat{k}}^{z}$;
- $H_{nonlocal}^{(2)}$: nonlocal two−spin terms, $J_{nm}^{zz}\sigma_n^{z}\sigma_m^{z}$;
- $H_{nonlocal}^{(M)}$: nonlocal multiple−spin terms, $J_{nm}^{\alpha\beta}\sigma_n^{\alpha}\left[\prod_{n<l<m}\sigma_l^{z}\right]\sigma_m^{\beta}$, etc.,

where $\alpha, \beta = x, y$, and $\hat{k} = \hat{1}, \cdots, \hat{d}$.

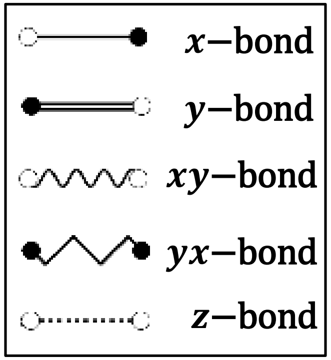
Construction rules

- **Firstly, divide the lattice into white (w) and black (b) sublattices arbitrary.**

Elementary rules:

① For a (local or nonlocal) link (n, m):
an x-bond is allocated for $n \in w$ and $m \in b$;
a y-bond is allocated for $n \in b$ and $m \in w$;
an xy-bond is allocated for $n \in w$ and $m \in w$;
a yx-bond is allocated for $n \in b$ and $m \in b$;

② Different z-bonds are not allowed to share the same site.



$\sigma_n^x \sigma_m^x$

$\sigma_n^y \sigma_m^y$

$\sigma_n^x \sigma_m^y$

$\sigma_n^y \sigma_m^x$

$\sigma_n^z \sigma_m^z$

beyond compass models

Construction rules

Exactly solvability: quadratic fermion terms

Jordan-Wigner transformation

$$\sigma_m^+ = c_m^\dagger e^{i\pi\left(\sum_{l<m}\hat{n}_l\right)},$$
$$\sigma_m^z = 2\hat{n}_m - 1,$$

Majorana fermion representation

$$n \in w,\ \eta_n = c_n^\dagger + c_n \text{ and } \gamma_n = i\left(c_n^\dagger - c_n\right)$$
$$n \in b,\ \eta_n = i\left(c_n^\dagger - c_n\right) \text{ and } \gamma_n = c_n^\dagger + c_n$$

$$J_{nm}^{\alpha\beta}\sigma_n^\alpha\left[\prod_{n<l<m}\sigma_l^z\right]\sigma_m^\beta$$

$$\sigma_{n\in w}^x\left[\prod_{n<l<m}(-\sigma_l^z)\right]\sigma_{m\in b}^x = -i\gamma_n\gamma_m,$$
$$\sigma_{n\in b}^y\left[\prod_{n<l<m}(-\sigma_l^z)\right]\sigma_{m\in w}^y = i\gamma_n\gamma_m,$$
$$\sigma_{n\in w}^x\left[\prod_{n<l<m}(-\sigma_l^z)\right]\sigma_{m\in w}^y = -i\gamma_n\gamma_m,$$
$$\sigma_{n\in b}^y\left[\prod_{n<l<m}(-\sigma_l^z)\right]\sigma_{m\in b}^x = i\gamma_n\gamma_m,$$

$$J_{n,n+\hat{1}}^{\alpha\beta}\sigma_n^\alpha\sigma_{n+\hat{1}}^\beta$$

$$\sigma_{n\in w}^x\sigma_{n+\hat{1}\in b}^x = -i\gamma_n\gamma_{n+\hat{1}},$$
$$\sigma_{n\in b}^y\sigma_{n+\hat{1}\in w}^y = i\gamma_n\gamma_{n+\hat{1}},$$
$$\sigma_{n\in w}^x\sigma_{n+\hat{1}\in w}^y = -i\gamma_n\gamma_{n+\hat{1}},$$
$$\sigma_{n\in b}^y\sigma_{n+\hat{1}\in b}^x = i\gamma_n\gamma_{n+\hat{1}},$$

All the possible quadratic γ−fermion terms by J−W transformation.

Construction rules

Exactly solvability: biquadratic fermion terms

$J_{nm}^{zz}\sigma_n^z\sigma_m^z$

$$\boxed{\sigma_n^z\sigma_m^z = i\hat{D}_{nm}\gamma_n\gamma_m}$$

Static Z_2 gauge field

$$\boxed{\hat{D}_{nm} = \pm i\eta_n\eta_m}$$

" $-$ ": n & m $\in$ the same sublattice
" $+$ ": n & m $\in$ opposite sublattice

Majorana fermion representation

$n \in w$, $\eta_n = c_n^\dagger + c_n$ and $\gamma_n = i\left(c_n^\dagger - c_n\right)$
$n \in b$, $\eta_n = i\left(c_n^\dagger - c_n\right)$ and $\gamma_n = c_n^\dagger + c_n$

- **Elementary rules:**

① …

② Different z-bonds are not allowed to share the same site.

$\left[\hat{D}_{nm}, \hat{D}_{n'm'}\right] = 0$, $\left[\hat{D}_{nm}, H\right] = 0$.
$\{D_{nm}\}$: a set of good quantum #s

$$\hat{D}_{nm}^2 = 1 \Rightarrow D_{nm} = \pm 1$$

- The eigenstates can be divided into different sectors according to $\{D_{nm}\}$.
- In each sector, allowed spin terms are trasformed to quadratic γ−fermion terms.

Construction rules

Separation of degrees of freedom

$$H = H_\gamma \otimes H_\eta$$

Majorana fermion representation

$$n \in w,\ \eta_n = c_n^\dagger + c_n \text{ and } \gamma_n = i\left(c_n^\dagger - c_n\right)$$
$$n \in b,\ \eta_n = i\left(c_n^\dagger - c_n\right) \text{ and } \gamma_n = c_n^\dagger + c_n$$

It is possible that some isolated η_n do not show up in H_η $\Rightarrow$ local degeneracy

To lift the local degeneracy: couple isolated η_n using nonlocal terms

quadratic η−fermion terms

$$\sigma^x_{n\in b}\left[\prod_{n<l<m}\left(-\sigma^z_l\right)\right]\sigma^x_{m\in w} = -i\eta_n\eta_m,$$
$$\sigma^y_{n\in w}\left[\prod_{n<l<m}\left(-\sigma^z_l\right)\right]\sigma^y_{m\in b} = i\eta_n\eta_m,$$
$$\sigma^x_{n\in b}\left[\prod_{n<l<m}\left(-\sigma^z_l\right)\right]\sigma^y_{m\in b} = -i\eta_n\eta_m,$$
$$\sigma^y_{n\in w}\left[\prod_{n<l<m}\left(-\sigma^z_l\right)\right]\sigma^x_{m\in w} = i\eta_n\eta_m,$$

$\Longleftrightarrow$

quadratic γ−fermion terms

$$\sigma^x_{n\in w}\left[\prod_{n<l<m}\left(-\sigma^z_l\right)\right]\sigma^x_{m\in b} = -i\gamma_n\gamma_m,$$
$$\sigma^y_{n\in b}\left[\prod_{n<l<m}\left(-\sigma^z_l\right)\right]\sigma^y_{m\in w} = i\gamma_n\gamma_m,$$
$$\sigma^x_{n\in w}\left[\prod_{n<l<m}\left(-\sigma^z_l\right)\right]\sigma^y_{m\in w} = -i\gamma_n\gamma_m,$$
$$\sigma^y_{n\in b}\left[\prod_{n<l<m}\left(-\sigma^z_l\right)\right]\sigma^x_{m\in b} = i\gamma_n\gamma_m,$$

Construction rules

Duality

$$w \Leftrightarrow b$$
$$\gamma \Leftrightarrow \eta$$

- A similar duality relates topological trivial and non-trivial phases in interacting Kitaev chains.
  J.J. Miao, H.K. Jin, F.C. Zhang, YZ (2017)

A way to construct new models

quadratic $\eta-$fermion terms

$$\sigma^x_{n\in b}\left[\prod_{n<l<m}(-\sigma^z_l)\right]\sigma^x_{m\in w} = -i\eta_n\eta_m,$$
$$\sigma^y_{n\in w}\left[\prod_{n<l<m}(-\sigma^z_l)\right]\sigma^y_{m\in b} = i\eta_n\eta_m,$$
$$\sigma^x_{n\in b}\left[\prod_{n<l<m}(-\sigma^z_l)\right]\sigma^y_{m\in b} = -i\eta_n\eta_m,$$
$$\sigma^y_{n\in w}\left[\prod_{n<l<m}(-\sigma^z_l)\right]\sigma^x_{m\in w} = i\eta_n\eta_m,$$

$\Longleftrightarrow$

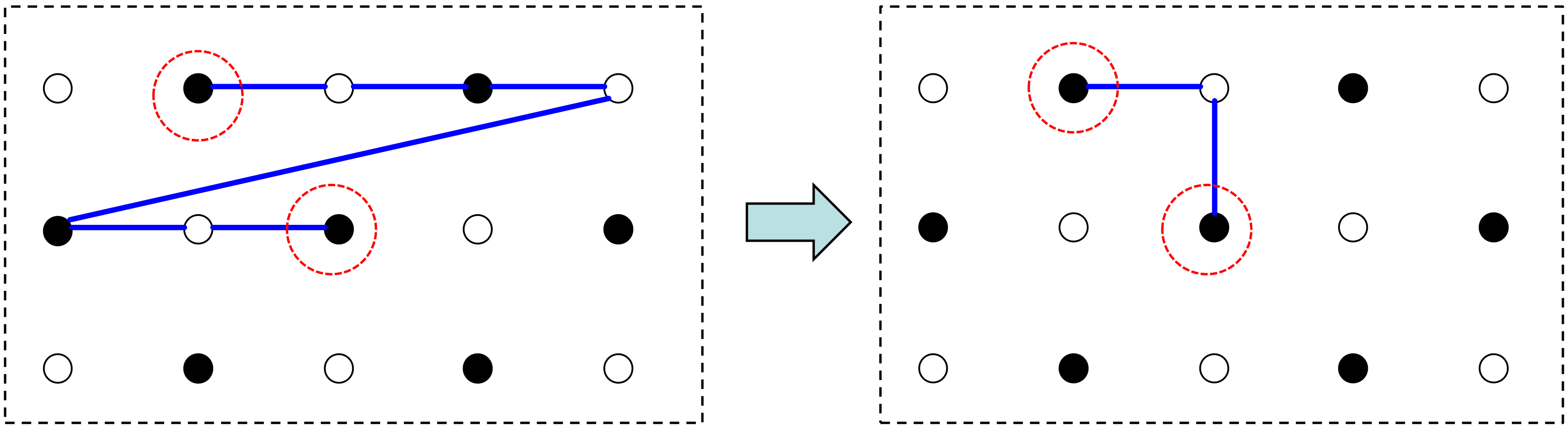
quadratic $\gamma-$fermion terms

$$\sigma^x_{n\in w}\left[\prod_{n<l<m}(-\sigma^z_l)\right]\sigma^x_{m\in b} = -i\gamma_n\gamma_m,$$
$$\sigma^y_{n\in b}\left[\prod_{n<l<m}(-\sigma^z_l)\right]\sigma^y_{m\in w} = i\gamma_n\gamma_m,$$
$$\sigma^x_{n\in w}\left[\prod_{n<l<m}(-\sigma^z_l)\right]\sigma^y_{m\in w} = -i\gamma_n\gamma_m,$$
$$\sigma^y_{n\in b}\left[\prod_{n<l<m}(-\sigma^z_l)\right]\sigma^x_{m\in b} = i\gamma_n\gamma_m,$$

Construction rules

Shortcut multiple-spin interactions

New multiple-spin interaction



$$i\sigma^x_n \sigma^x_{n+\hat{1}} \sigma^z_{n+\hat{1}} \sigma^z_{n+\hat{1}+\hat{2}} = \sigma^x_n \sigma^y_{n+\hat{1}} \sigma^z_{n+\hat{1}+\hat{2}}$$

$$\begin{aligned} & \gamma_n \gamma_{n+\hat{1}} i\hat{D}_{n+\hat{1},n+\hat{1}+\hat{2}} \gamma_{n+\hat{1}} \gamma_{n+\hat{1}+\hat{2}} \\ = \; & i\hat{D}_{n+\hat{1},n+\hat{1}+\hat{2}} \gamma_n \gamma_{n+\hat{1}+\hat{2}} \end{aligned}$$

Construction rules

Supplementary rules:

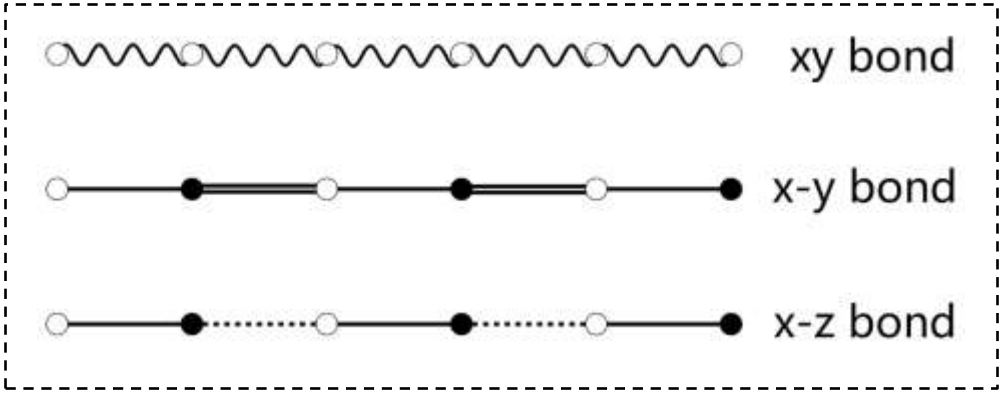
① To add η-fermion quadratic terms using a nonlocal link (n, m): n and m are not allowed to coincide with site connected by existing z-bonds in the original Hamiltonian constructed subjected to the two elementary rules.

② To add shortcut multiple-spin interactions: for a step along the $\hat{1}$-direction, the two-spin term should be $\sigma_l^{\alpha}\sigma_{l+\hat{1}}^{\beta}$ with $\alpha, \beta = x, y$; for a step along the other directions, the two-spin terms should be $\sigma_l^{z}\sigma_{l+\delta}^{z}$ with $\delta \neq \hat{1}$, and there must exist a local z-bond on this step in the original Hamiltonian.

③ In the above, the indices α and β should be chosen as follows: for $l \in w$ and $l + \hat{1} \in b$, $(\alpha, \beta) = (x, x)$; for $l \in b$ and $l + \hat{1} \in w$, $(\alpha, \beta) = (y, y)$; for $l \in w$ and $l + \hat{1} \in w$, $(\alpha, \beta) = (x, y)$; for $l \in b$ and $l + \hat{1} \in b$, $(\alpha, \beta) = (y, x)$.

Generating new models: 1D examples

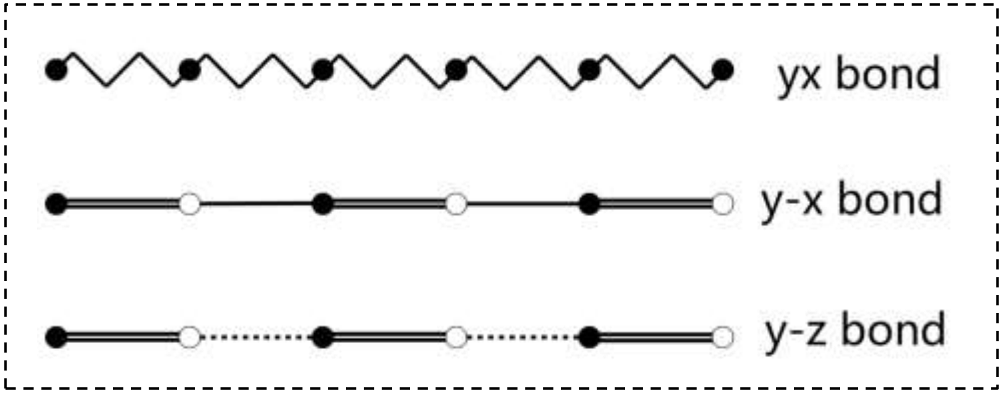
Three parent models in 1D

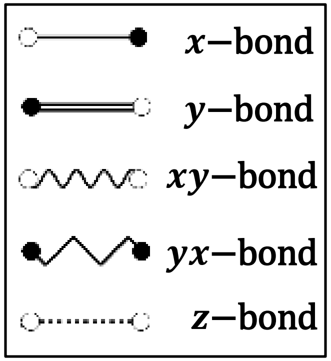


(1) duality

$$w \Leftrightarrow b$$
$$\gamma \Leftrightarrow \eta$$

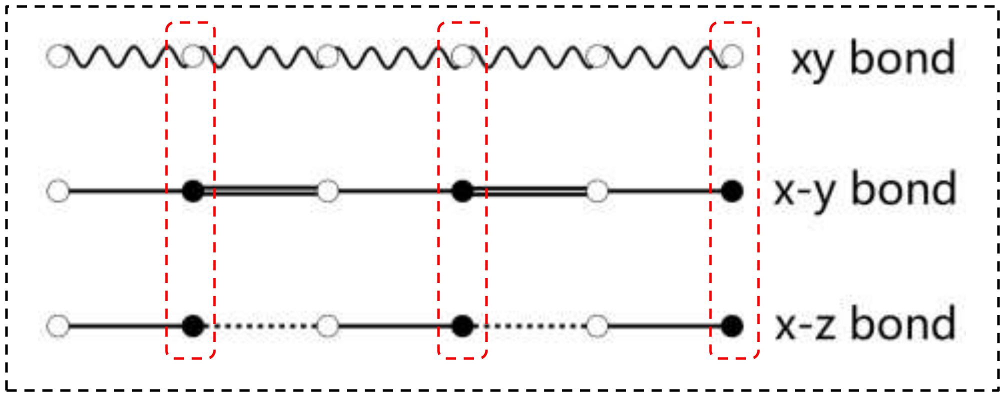
Dual models in 1D





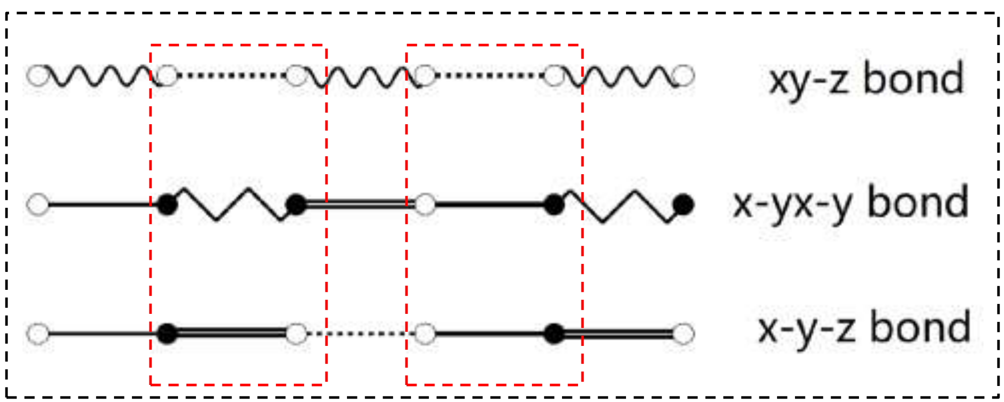
Generating new models: 1D examples

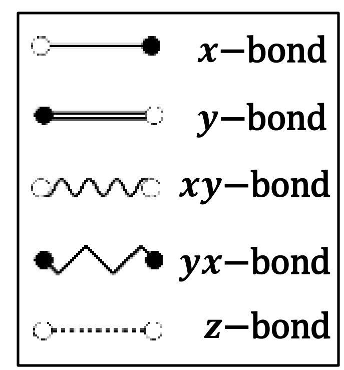
Three parent models in 1D



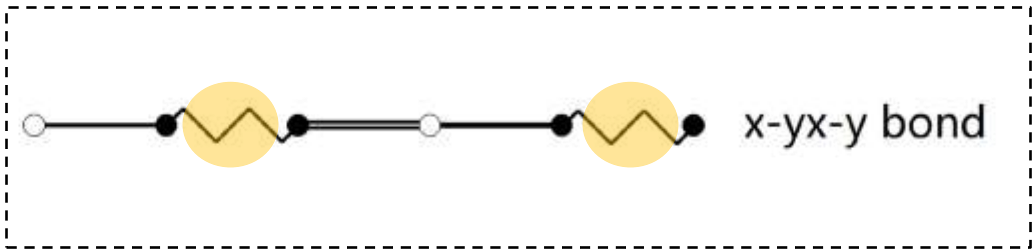
(2) split one site and insert a local bond

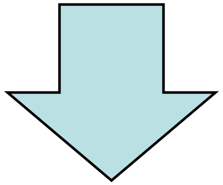
Models with enlarged unit cell



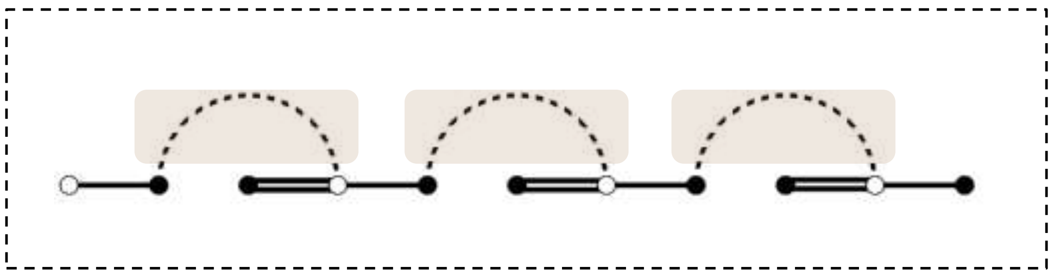


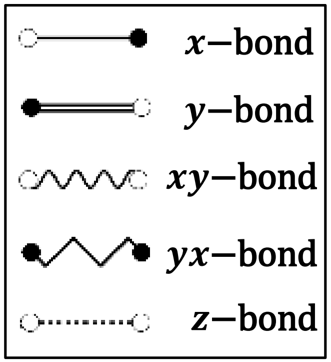
Generating new models: 1D examples





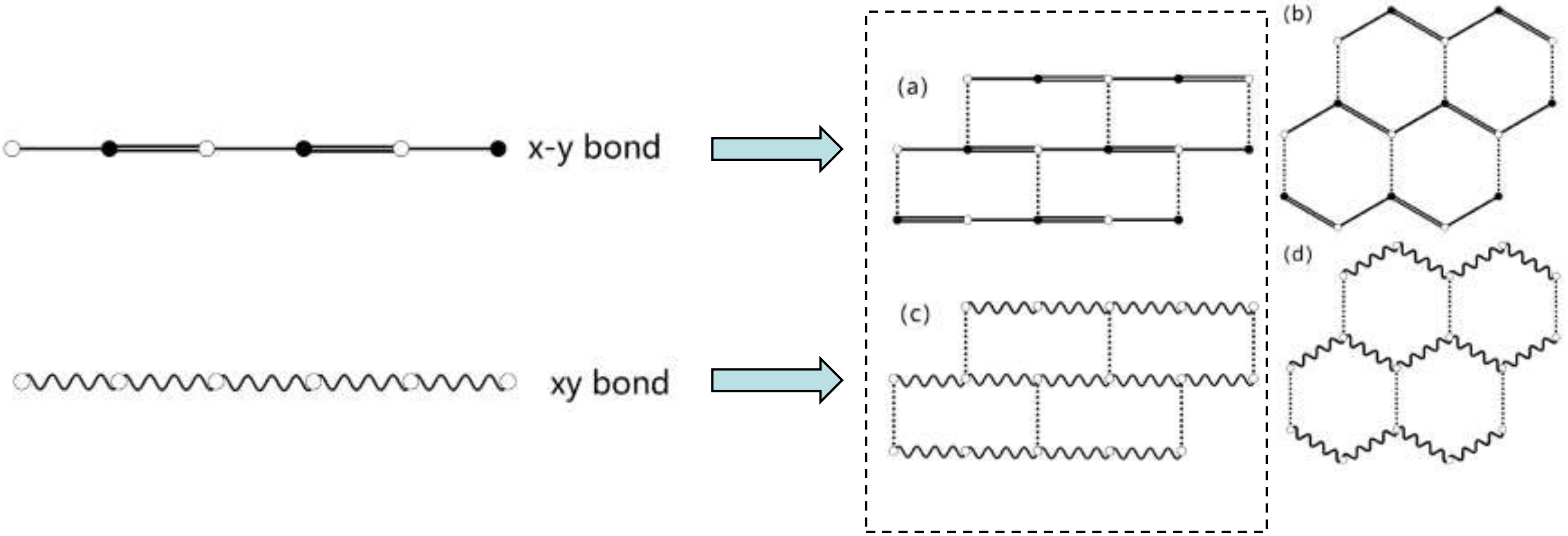
(3) erase bonds and add nonlocal bonds



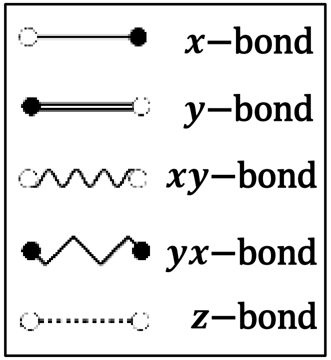


Generating new models: from 1D to 2D

Two parent models in 2D

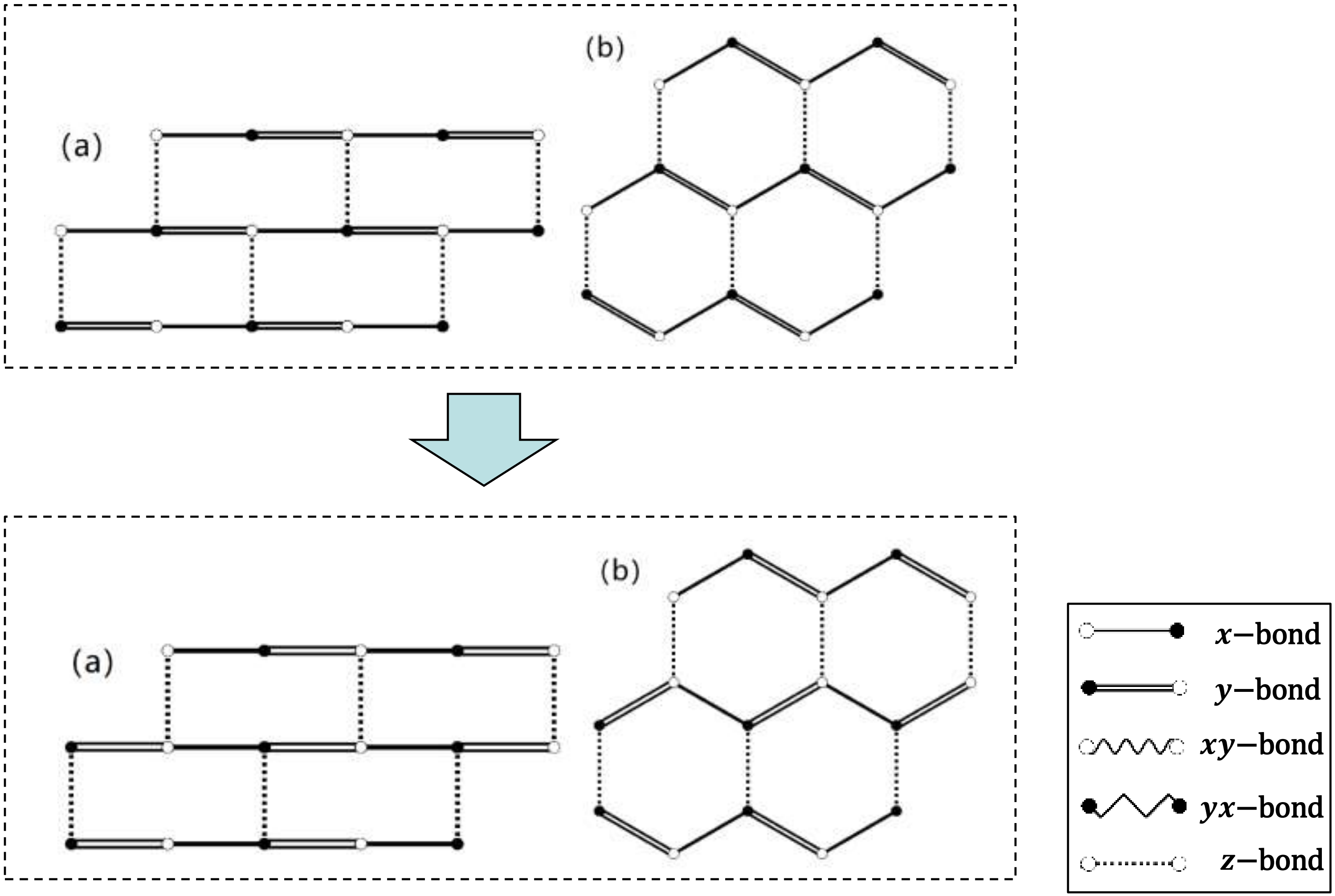


couple through z–bond



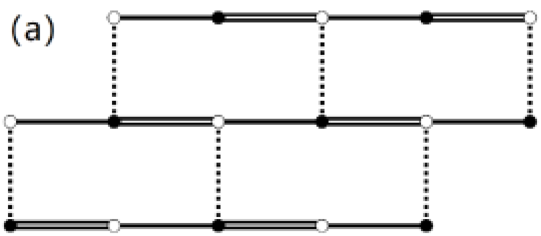
Generating new models: 2D examples

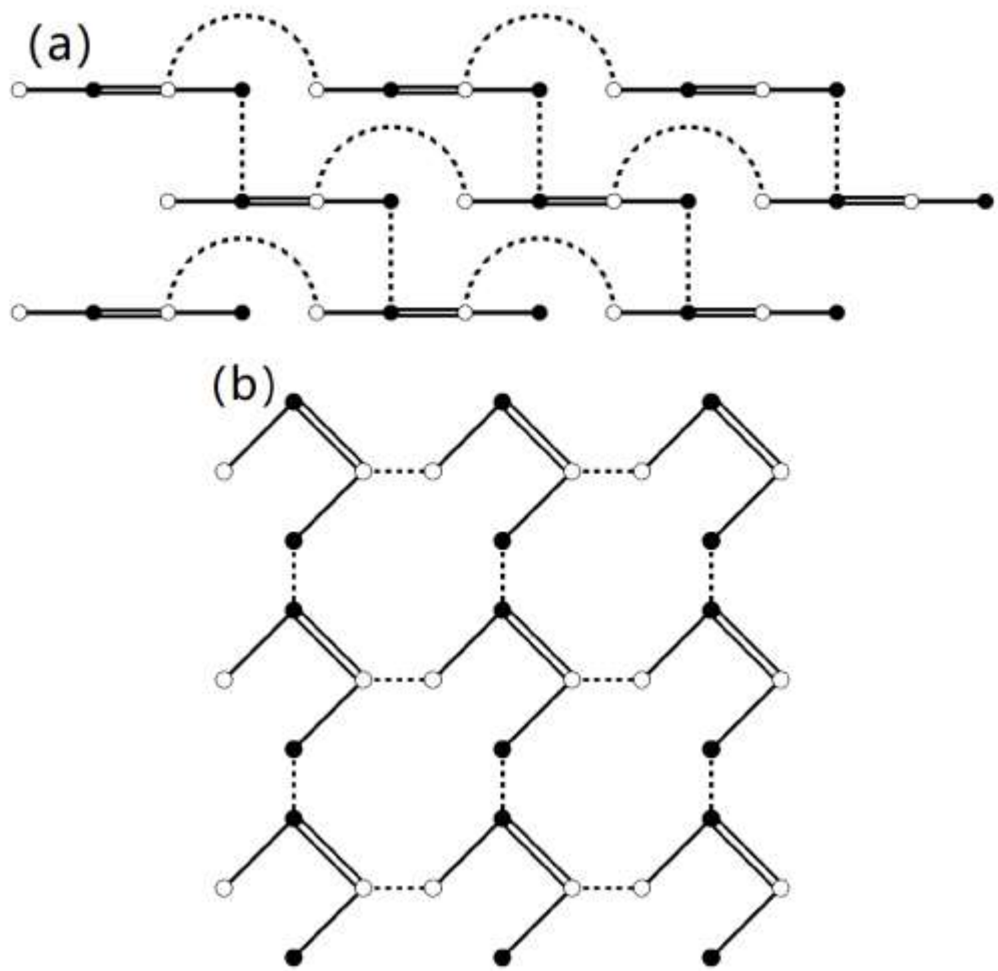
Duality transformation can be performed along each chain independently.

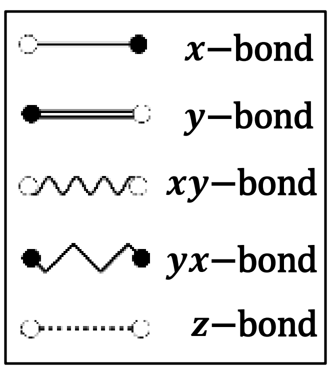


Generating new models: 2D examples

Split sites and insert nonlocal bonds

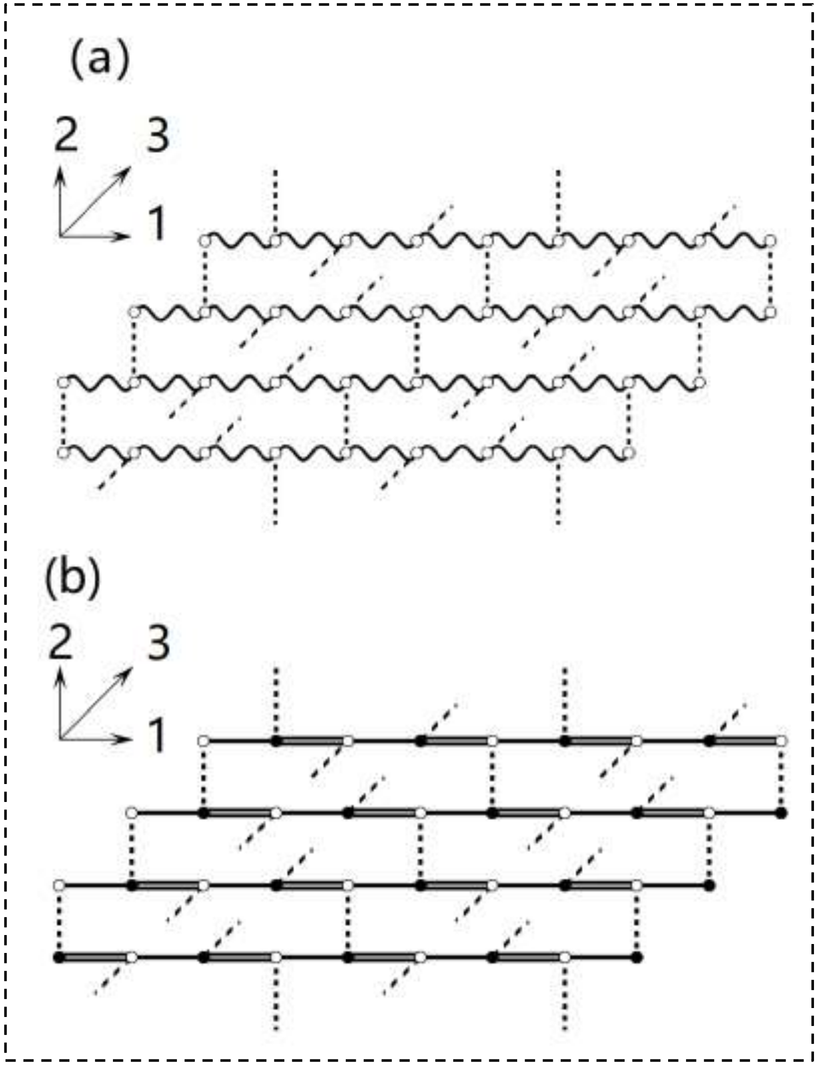




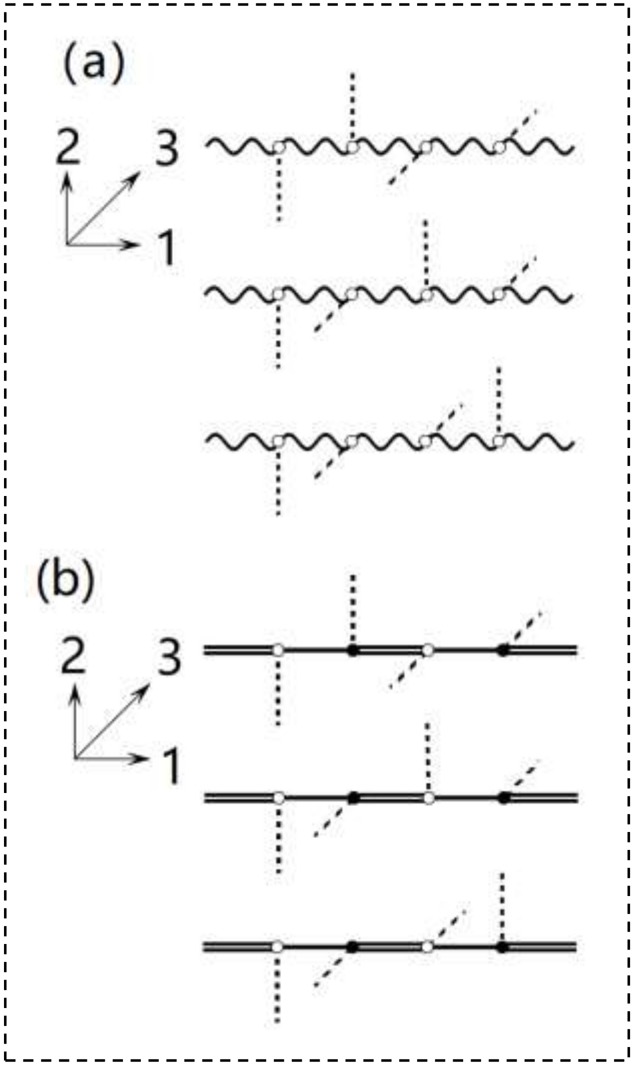


Generating new models: 3D examples

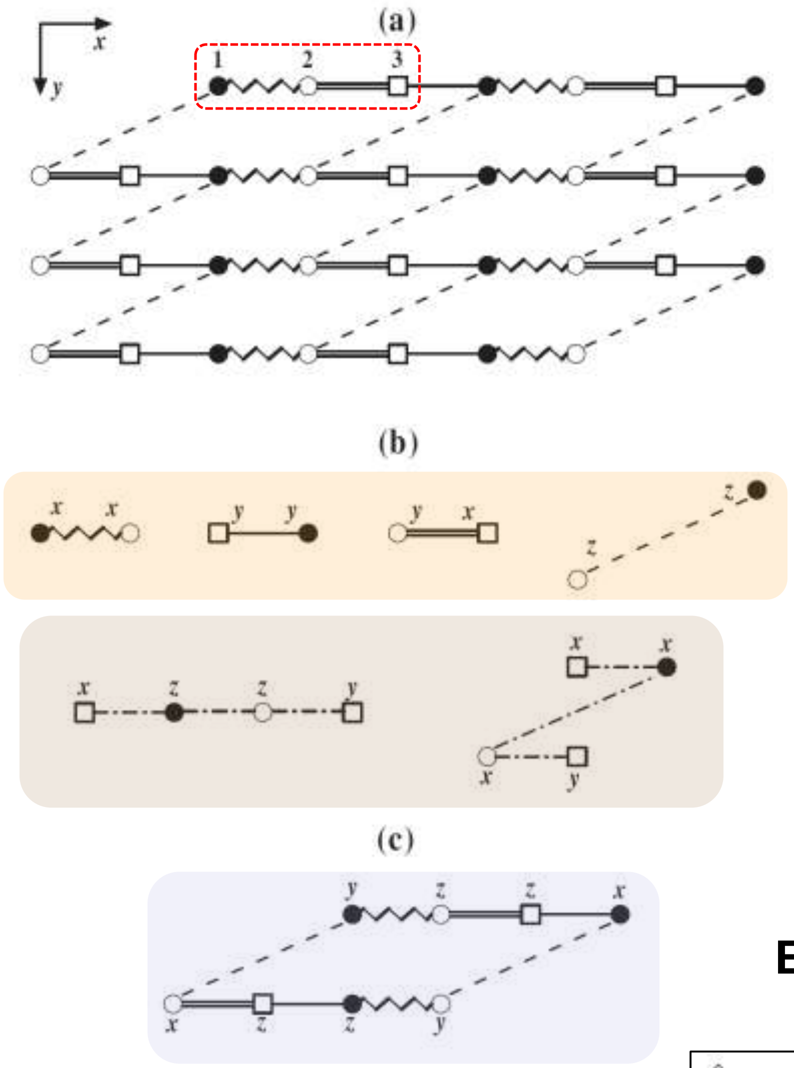
Two parent models in 3D



Three types of unit cells



2D example: a Mott insulator model



$$
\begin{aligned}
H &= H_0 + H_1, && \text{(1a)}\\
H_0 &= \sum_{\vec{r}} J_x \sigma^x_{\vec{r},1}\sigma^x_{\vec{r},2} + J_{yx}\sigma^y_{\vec{r},2}\sigma^x_{\vec{r},3} + J_y \sigma^y_{\vec{r},3}\sigma^y_{\vec{r}+\hat{x},1} \\
&\quad + J_z \sigma^z_{\vec{r},1}\sigma^z_{\vec{r}-\hat{x}+\hat{y},2}, && \text{(1b)}\\
H_1 &= \sum_{\vec{r}} t_x \sigma^x_{\vec{r},3}\sigma^z_{\vec{r}+\hat{x},1}\sigma^z_{\vec{r}+\hat{x},2}\sigma^y_{\vec{r}+\hat{x},3} \\
&\quad + t_y \sigma^x_{\vec{r},3}\sigma^x_{\vec{r}+\hat{x},1}\sigma^x_{\vec{r}+\hat{y},2}\sigma^y_{\vec{r}+\hat{y},3}, && \text{(1c)}
\end{aligned}
$$

H_0: two−spin interactions

H_1: four−spin interactions
→ lift local degeneracy

Elementary plaquette & Flux operator

$$\hat{\phi}_p = -\sigma^y_{\vec{r},1}\sigma^z_{\vec{r},2}\sigma^z_{\vec{r},3}\sigma^x_{\vec{r}+\hat{x},1}\sigma^y_{\vec{r}+\hat{y},2}\sigma^z_{\vec{r}+\hat{y},1}\sigma^z_{\vec{r}-\hat{x}+\hat{y},3}\sigma^x_{\vec{r}-\hat{x}+\hat{y},2}$$

2D example: a Mott insulator model

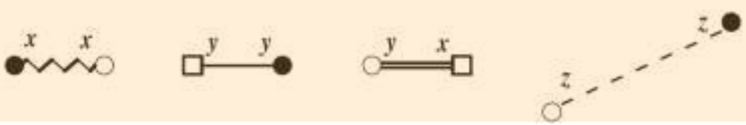
Majorana representation

● $\eta_n = c_n^\dagger + c_n$ and $\beta_n = i\left(c_n^\dagger - c_n\right)$

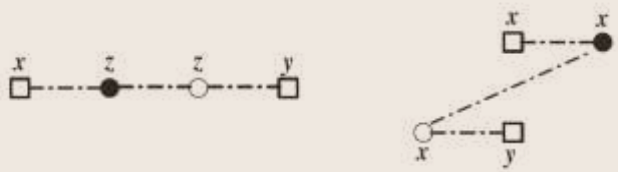
□ ○ $\eta_n = i\left(c_n^\dagger - c_n\right)$ and $\beta_n = c_n^\dagger + c_n$

Jordan-Wigner transformation

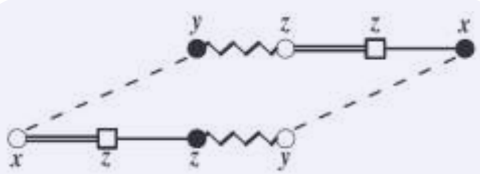
Static Z_2 gauge field: $\hat{D}_{\vec{r}} = i\eta_{\vec{r},1}\eta_{\vec{r}-\hat{x}+\hat{y},2}$



$$
\begin{aligned}
\sigma^x_{\vec{r},1}\sigma^x_{\vec{r},2} &= -i\beta_{\vec{r},1}\beta_{\vec{r},2}, \\
\sigma^y_{\vec{r},2}\sigma^x_{\vec{r},3} &= -i\beta_{\vec{r},2}\beta_{\vec{r},3}, \\
\sigma^y_{\vec{r},3}\sigma^y_{\vec{r}+\hat{x},1} &= i\beta_{\vec{r},3}\beta_{\vec{r}+\hat{x},1}, \\
\sigma^z_{\vec{r},1}\sigma^z_{\vec{r}-\hat{x}+\hat{y},2} &= i\hat{D}_{\vec{r}}\beta_{\vec{r},1}\beta_{\vec{r}-\hat{x}+\hat{y},2},
\end{aligned}
$$



$$
\begin{aligned}
\sigma^x_{\vec{r},3}\sigma^z_{\vec{r}+\hat{x},1}\sigma^z_{\vec{r}+\hat{x},2}\sigma^y_{\vec{r}+\hat{x},3} &= i\eta_{\vec{r},3}\eta_{\vec{r}+\hat{x},3}, \\
\sigma^x_{\vec{r},3}\sigma^x_{\vec{r}+\hat{x},1}\sigma^x_{\vec{r}+\hat{y},2}\sigma^y_{\vec{r}+\hat{y},3} &= -i\hat{D}_{\vec{r}+\hat{x}}\eta_{\vec{r},3}\eta_{\vec{r}+\hat{y},3},
\end{aligned}
$$



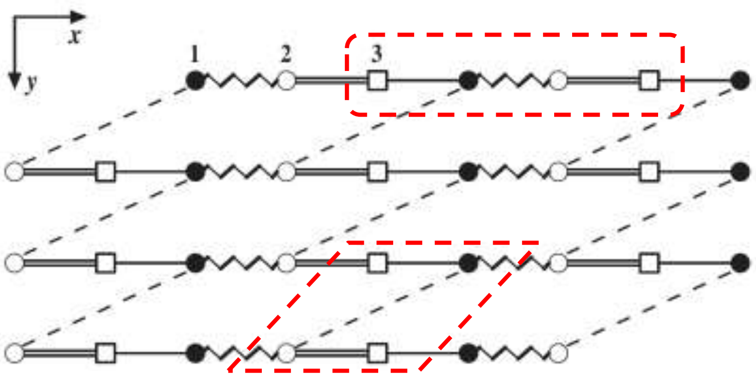
$$\hat{\phi}_p = \hat{D}_{\vec{r}}\hat{D}_{\vec{r}+\hat{x}}$$

2D example: a Mott insulator model

Separation of degrees of freedom

$$
\begin{aligned}
H &= H_0 + H_1 \\
&= H_\beta \otimes H_D + H_{\eta_3} \otimes H_D
\end{aligned}
$$

$$\hat{D}_{\vec{r}} = i\eta_{\vec{r},1}\eta_{\vec{r}-\hat{x}+\hat{y},2}$$



Majorana representation

● $\eta_n = c_n^\dagger + c_n$ and $\beta_n = i\left(c_n^\dagger - c_n\right)$

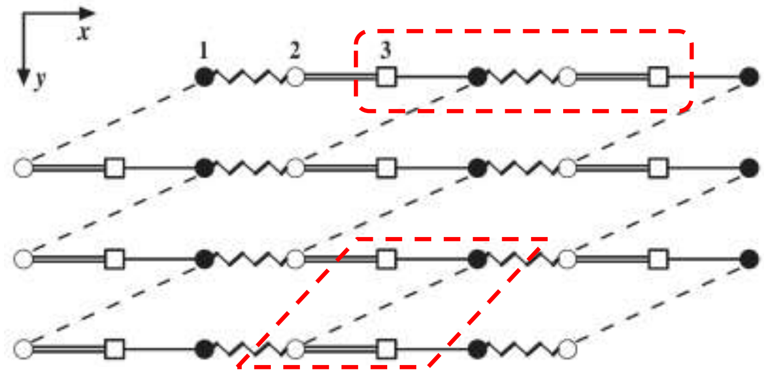
□ ○ $\eta_n = i\left(c_n^\dagger - c_n\right)$ and $\beta_n = c_n^\dagger + c_n$

Exact solvability: Given $\{D_{\vec{r}} = \pm 1\} \rightarrow$ Both H_0 and H_1 are quadratic form.

H_0: Absence of $\eta_3 \rightarrow 2^{L_x L_y/2}$—fold degeneracy
H_1: Lift the local degeneracy

2D example: a Mott insulator model

Lift the local degeneracy

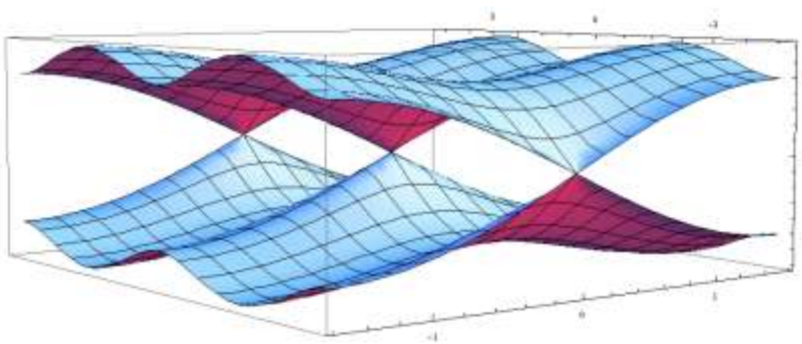


H_1: Free Majorana fermions η_3 on a square lattice, coupled to a static Z_2 gaugefield $D_{\vec{r}}$

Ground state: π − flux state, $\phi_p = -1$, on every plaquette

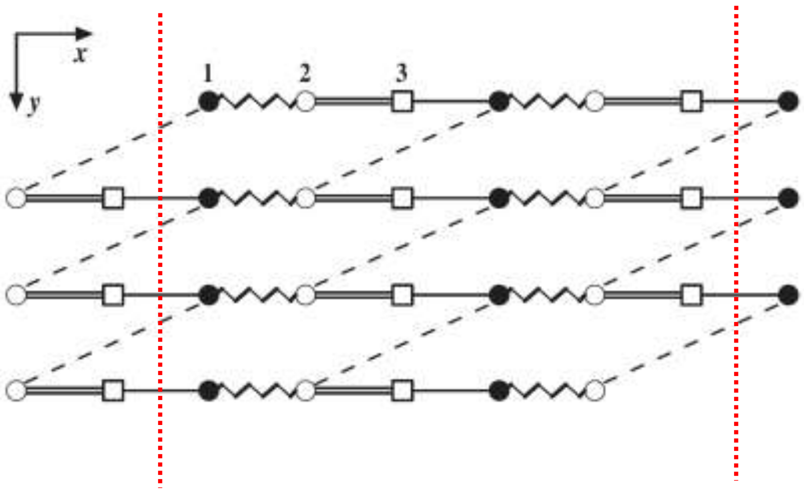
Energy dispersion: π − flux state

$$\epsilon_{\eta_3}(\vec{k}) = \pm\sqrt{(t_x \sin k_x)^2 + (t_y \sin k_y)^2}$$



2D example: a Mott insulator model

Boundary conditions: open BC vs. periodic BC



Boundary terms — JW transformation

$$\sigma^y_{L_x,n_y,3}\sigma^y_{1,n_y,1} = i\beta_{L_x,n_y,3}\beta_{1,n_y,1}\hat{F}_{n_y}$$

$$\hat{F}_{n_y} = e^{i\pi\hat{N}_{n_y}} \quad \hat{N}_{n_y} = \sum_{n_x,\mu}\hat{n}_{n_x,n_y,\mu}$$

Fluxes on edge plaquettes

$$\hat{\phi}_p = \hat{D}_{L_x,n_y}\hat{D}_{1,n_y}\hat{F}_{n_y}$$

$$\hat{\phi}_p = \hat{D}_{1,n_y}\hat{D}_{2,n_y}\hat{F}_{n_y+1}$$

Open boundary condition

- Good quantum #s: $D_{\vec{r}}$
- 2^{L_y}-fold degeneracy: Majorana zero modes at edges

Periodic boundary condition

- Good quantum #s: $\{\phi_p, \Phi_x, \Phi_y\}$
- Z_2 global fluxes: Φ_x, Φ_y

$$\hat{\Phi}_x = \hat{F}_{n_y=1} \quad \hat{\Phi}_y = \prod_{n_y}\hat{D}_{\vec{r}=(1,n_y)}$$

2D example: a Mott insulator model

Degrees of freedom: a $3 \times L_x \times L_y$ lattice

- Possible spin states: $2^{3L_xL_y}$
- Possible fermion states: $2^{3L_xL_y+1}$
  - $\{\phi_p, \Phi_x, \Phi_y\}$: $2^{L_xL_y+1}$
  - $\{\eta_{\vec{r},3}, \beta_{\vec{r},1}, \beta_{\vec{r},2}, \beta_{\vec{r},3}\}$: $2^{2L_xL_y}$

Half of the states in the fermion representation are unphysical.

Origin: $\{\phi_p, \Phi_x, \Phi_y\}$ is presumed.

Projection: to remove the unphysical states

$$\boxed{\hat{P} = (1 + F\hat{F})/2}$$

total fermion # parity: $\hat{F} = \prod_{n_y} \hat{F}_{n_y}$

Deductions:

① For a given set of $\{\phi_p, \Phi_x, \Phi_y\}$, the projection $\hat{P}$ survives half fermionic states with compatible F.

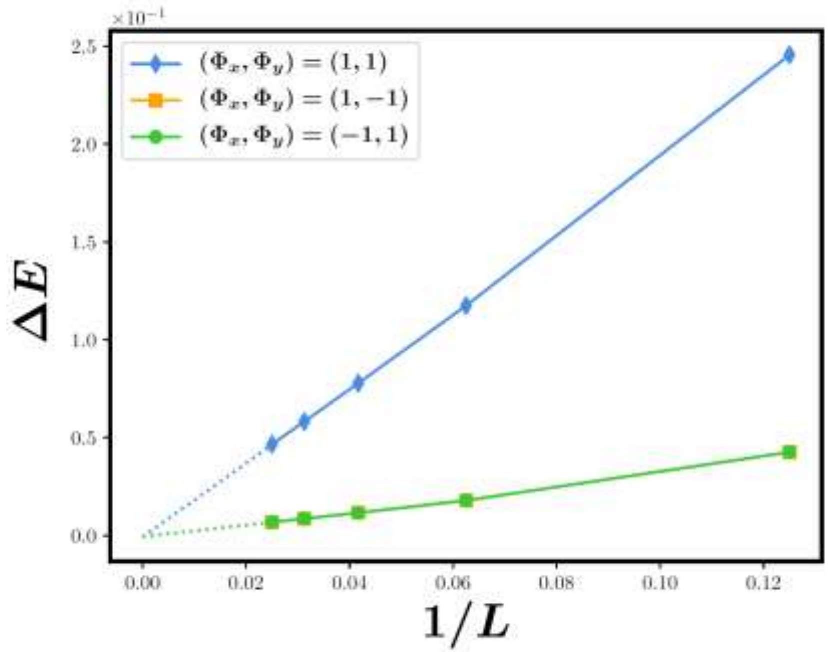
② A physical spin excitation should be composed of even number of fermions.

2D example: a Mott insulator model

Ground states: topological degeneracy

- Ground states: $\boldsymbol{\pi}-\textbf{flux states}$
- Unprojected degenerate ground states:

  $\Phi_x = \pm 1, \Phi_y = \pm 1$
- Topological degeneracy: $\Delta E \propto 1/L$
- Projection: survives 3 ground states
  - $\hat{P}|G\rangle_{\Phi_x=\Phi_y=1} = 0$ for $L_x\,, L_y = even$
  - Pairing terms vanish at $q_x = q_y = 0$
- Robust against disorders

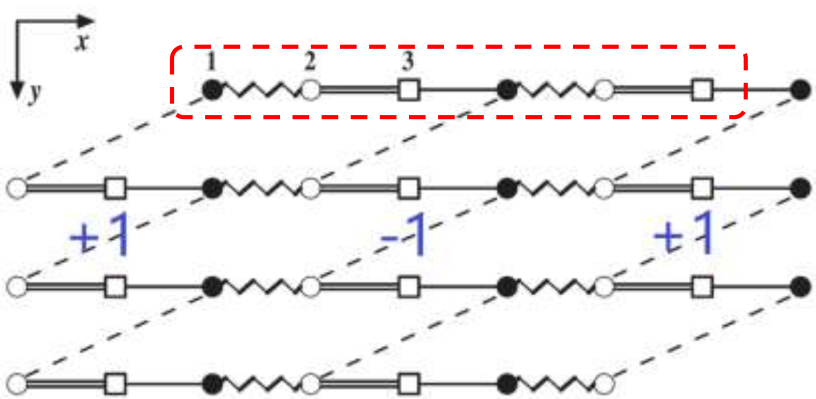


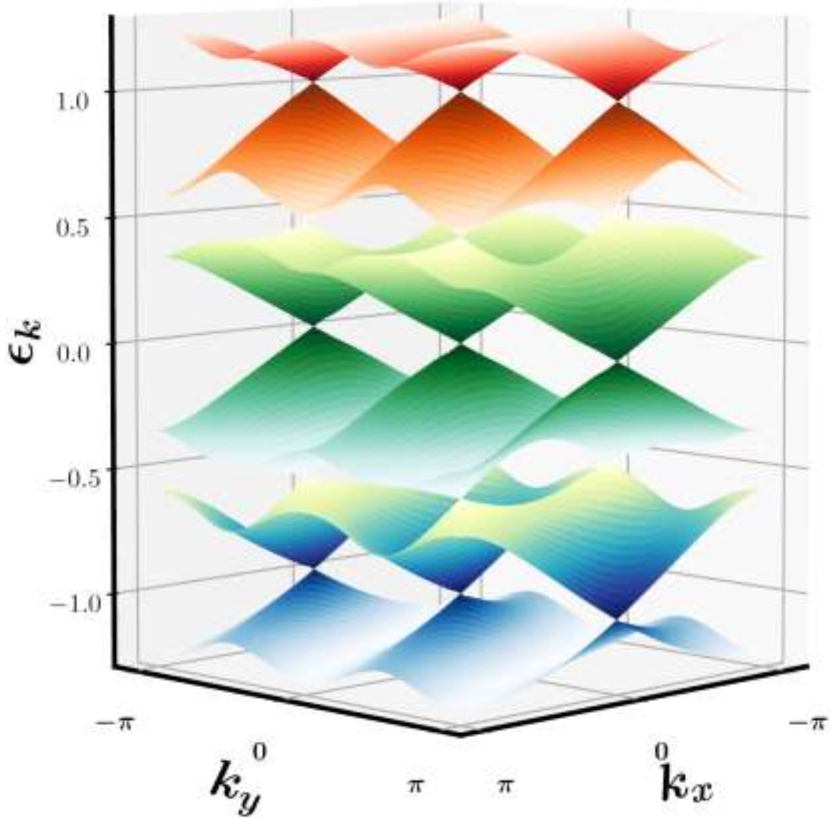
3-fold topological degeneracy on a torus

2D example: a Mott insulator model

Bulk spinon excitations

- ☐ $\pi -$ **flux states**: magnetic unit cell
  - ☐ 6 sites in each magnetic unit cell
- ☐ Six bands for β-Majorana fermions
- ☐ Two point nodes: $(0,0)$ and $(0, \pi)$
- ☐ Dirac-like dispersion around nodes





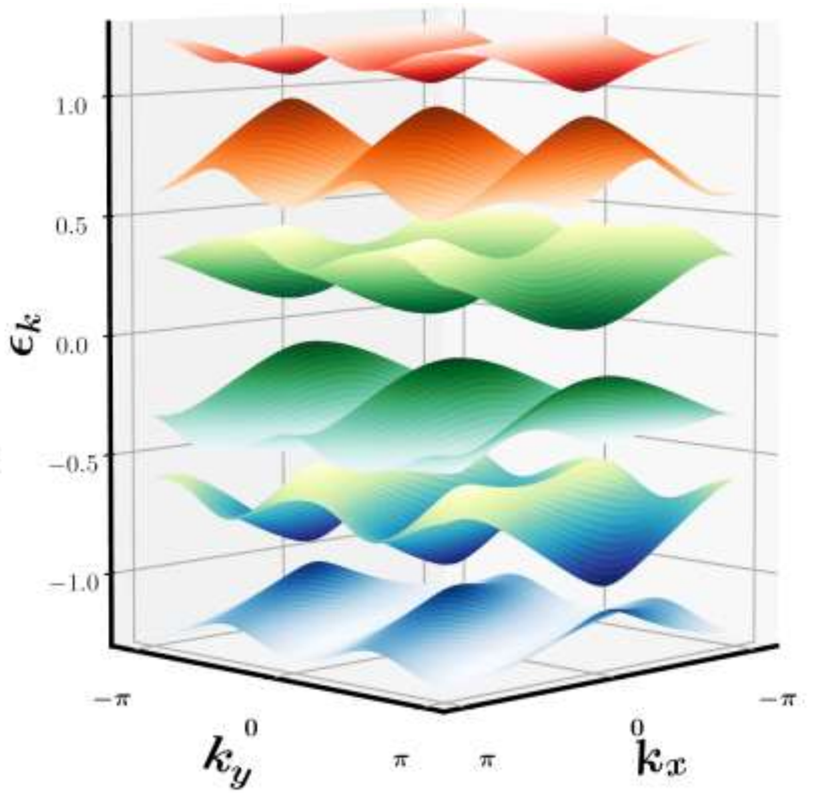
2D example: a Mott insulator model

Breaking time-reversal symmetry (TRS)

- Magnetic field $\sum_n \vec{h} \cdot \vec{\sigma}_n$

- 3^{rd} order perturbation: exactly solvable

$$
\begin{aligned}
H' = \frac{h^3}{\Delta_v^2} \sum_{\vec{r}} & \sigma^x_{\vec{r},1}\sigma^z_{\vec{r},2}\sigma^x_{\vec{r},3} + \sigma^y_{\vec{r},2}\sigma^z_{\vec{r},3}\sigma^y_{\vec{r}+\hat{x},1} \\
& +\sigma^y_{\vec{r},3}\sigma^z_{\vec{r}+\hat{x},1}\sigma^x_{\vec{r}+\hat{x},2} + \sigma^z_{\vec{r},1}\sigma^x_{\vec{r}-\hat{x}+\hat{y},1}\sigma^y_{\vec{r}-\hat{x}+\hat{y},2} \\
& +\sigma^y_{\vec{r},1}\sigma^x_{\vec{r},2}\sigma^z_{\vec{r}-\hat{x}+\hat{y},2} + \sigma^y_{\vec{r},3}\sigma^x_{\vec{r}+\hat{x},1}\sigma^z_{\vec{r}+\hat{y},2} \\
& +\sigma^z_{\vec{r},1}\sigma^x_{\vec{r}-\hat{x}+\hat{y},2}\sigma^x_{\vec{r}-\hat{x}+\hat{y},3},
\end{aligned}
$$



- Chern numbers

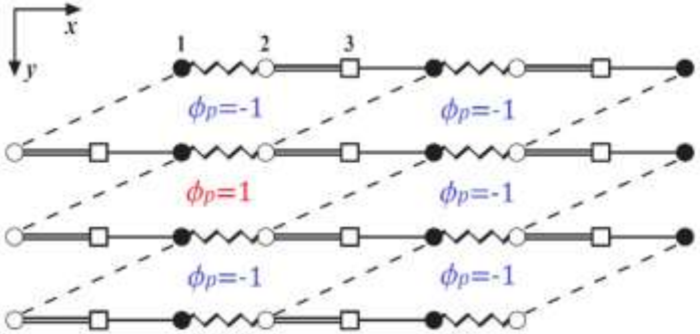
$$C\# = (-1, -1, 1, -1, 1, 1)$$

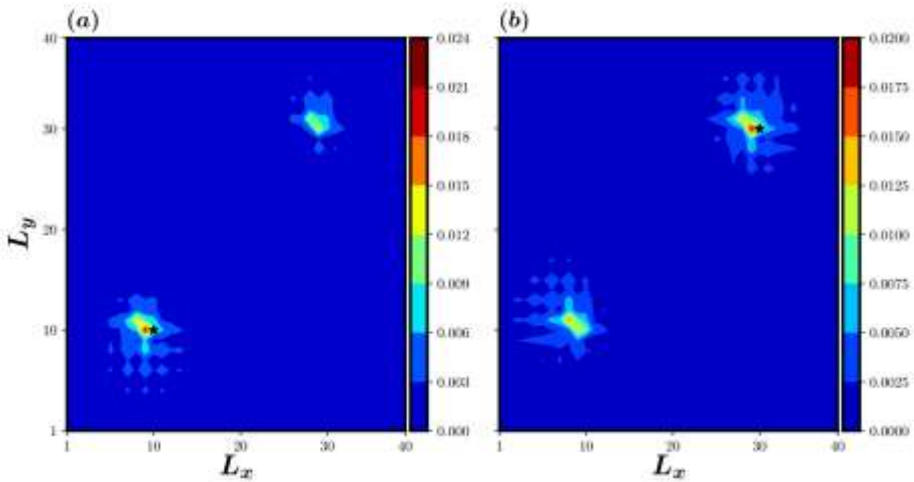
- 5^{th} order perturbation: open a gap for η_3 MFs

2D example: a Mott insulator model

Breaking time-reversal symmetry (TRS)

- Z_2 vortices
  - PBC: even # of vortices
- A pair of vortices
  - One Majorana zero mode (MZM) in each vortex core center
  - Extra double degeneracy due to MZMs?
  - MZM changes Fermion # parity → Projection removes half states.
  - 4-fold GS degeneracy regarding global fluxes Φ_x and Φ_y
- $2n$ well-separated vortices
  - 2^{n+1}-fold degeneracy





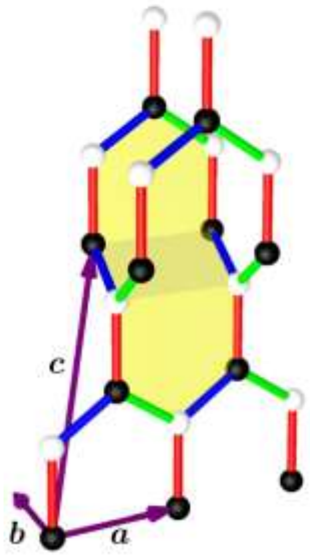
2D example: a Mott insulator model

Summary

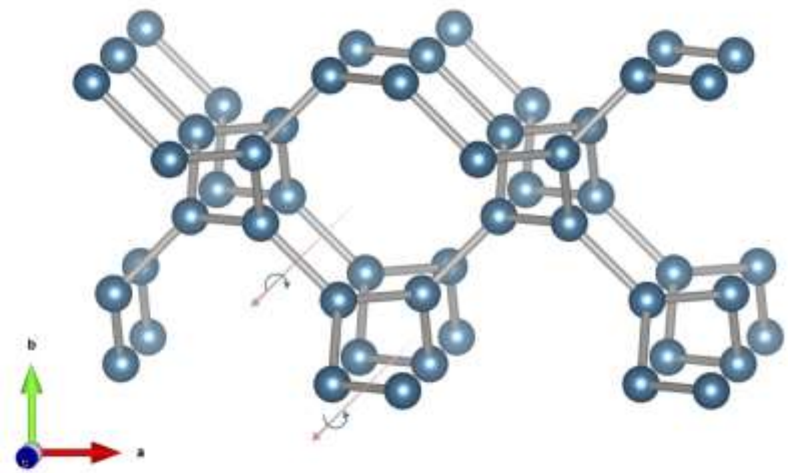
- Mott insulator model: odd number of spin-1/2 per unit cell.
- Algebraic quantum spin liquid ground state.
- Ground states are of three-fold topological degeneracy.
- Bulk spinon excitations: two Dirac nodes.
- Breaking TRS
  - Topologically nontrivial spinon bands: odd Chern numbers.
  - Z_2 vortices obey non-Abelian statistics.

More models in 3D

Generate new models from an existing model.



hyperhoneycomb

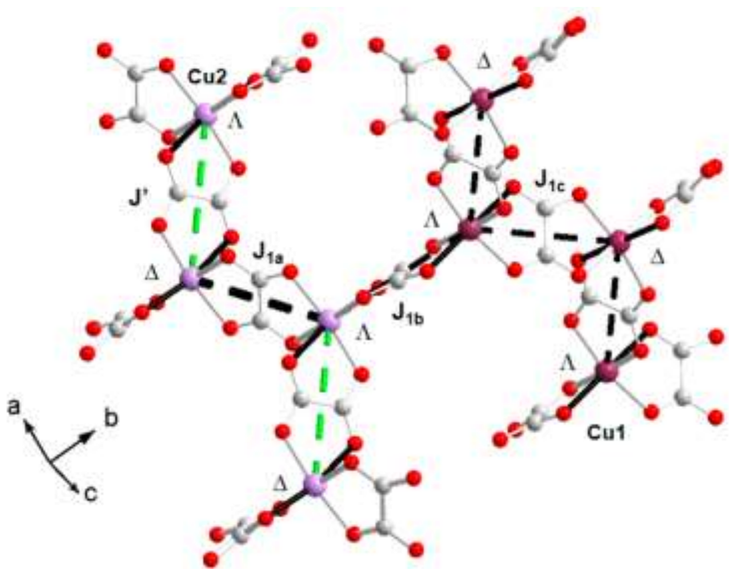


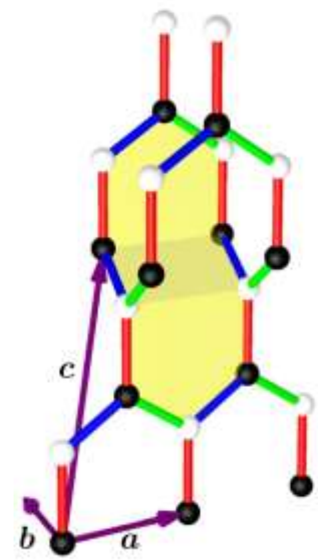
hyperoctagon

Si, Yu (2007); Ryu (2009); Mandal, Surendran (2009); Kimchi, Analytis, Vishwanath (2014); Nasu, Udagawa, Motome (2014); Hermanns, O'Brien, Trebst (2015); Hermanns, Trebst (2016)

Possible material realization

Metal organic framework (MOF)





Hyerhoneycomb: Cu-network

Zhang, Baker, …, Pratt, et. al. (2018)

Summary

- Construct a class of generalized Kitaev spin-1/2 models in arbitrary dimensions
  - Beyond the category of quantum compass models
- Provide some methods to generate new models from existing models.
- A particular 2D example: Pristine Mott insulator.

Thank you for attention