Discovery of the chiral Majorana fermion and its application to quantum computing

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Dirac equation and the anti-particle

In 1928, Dirac unified Einstein’s special theory of relativity with quantum mechanics, and introduced the Dirac equation:

\[(i\gamma^\mu \partial_\mu - m)\psi = 0\]

Dirac equation gives negative energy solutions, which led Dirac to predict the existence of anti-particle.

In 1932, the positron, the anti-particle of the electron was discovered by CD Anderson in cosmic rays.
Majorana and his fermion

In 1937, Ettore Majorana asked the question: can fermions be their own antiparticles?

The **Dirac equation** is known to describe charged fermions:

\[(i\gamma^\mu\partial_\mu - m)\psi = 0\]

where \(\gamma^\mu\) are Dirac’s anticommuting **Gamma matrices**.

Majorana claimed if all \(\gamma^\mu\) are selected imaginary, one can make \(\psi\) real, describing a charge neutral, spin \(\frac{1}{2}\) fermion being its own antiparticle, i.e., **Majorana fermion**, obeying **Majorana equation**.

\[
\begin{align*}
\tilde{\gamma}^0 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \\
\tilde{\gamma}^1 &= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \\
\tilde{\gamma}^2 &= \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \\
\tilde{\gamma}^3 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}
\end{align*}
\]

Gamma matrices in **Majorana equation**
Properties of the Majorana fermion

Neutrino could be a Majorana fermion, with Majorana mass term.

Majorana fermion is essential for supersymmetry.

Chiral Majorana fermion could exist in 1+1 and 9+1 dimensions, both essential for the superstring theory.

Majorana fermion could arise as quasi-particles of topological states of quantum matter.

Majorana fermion could be used for topological quantum computing.

Search for hypothetical particles/waves

Higgs boson, gravitational wave
Majorana fermion
Magnetic monopole
Axion
Dark matter particle
The race for Majorana fermion

Majorana returns

Frank Wilczek

In his short career, Ettore Majorana made several profound contributions. One of them, his concept of ‘Majorana fermions’ — particles that are their own antiparticle — is finding ever wider relevance in modern physics.

Viewpoint

Race for Majorana fermions

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The race for realizing Majorana fermions—elusive particles that act as their own antiparticles—heats up, but we still await ideal materials to work with.
Topological insulators and superconductors

Full pairing gap in the bulk, gapless Majorana edge and surface states

Chiral Majorana fermions

Chiral SC

massless Majorana fermions

(Qi, Hughes, Raghu and Zhang, PRL, 2009)
Quantum Spin Hall Effect and Topological Phase Transition in HgTe Quantum Wells

B. Andrei Bernevig,¹,² Taylor L. Hughes,¹ Shou-Cheng Zhang¹*

All topological insulators are predicted based on the principle of band inversion.
Experimental observation of the QSH edge state
(Konig et al, Science 2007)
Gapped Dirac fermions on the surface, chiral fermions on the domain wall

QAH can be realized in magnetic TI (Qi, Hughes, Zhang, PRB 2008)
Evolution from QSH to QAH

Helical edge states of QSH protected by time reversal symmetry

Magnetic order breaks time reversal symmetry, removes the degeneracy

In the extreme limit, only chiral edge states of the QAH state remains
Discovery of the QAH (Science 340, 167 (2013))
Majorana zero mode

Majorana zero mode (MZM) is predicted to appear
• on the 1D TSC boundary:

\[ \frac{dI}{dV} \propto \text{Density of States} \]

which should exhibit a zero bias peak.

• or in the vortex of a 2D chiral TSC:

Detection method: STM tunneling

Kitaev 2000, Lutchyn, Sau, Das Sarma, 2010; Oreg, Refael, von Oppen, 2010

Moore, Read 2000, Sau et. al., 2010; Fu, Kane, 2010, Alicea 2010, Qi, Hughes, Zhang, 2010
Quantized Zero Bias Peak in Theory

For STM tip solely coupled to a MZM (coupling strength $\Gamma$):

**charge neutrality** $\rightarrow$ resonant Andreev amplitude $R_A = 1$

\[
\frac{dI}{dV} = 2R_A = \frac{2e^2}{h} \text{ quantized at zero energy.}
\]

- The quantization is lost when the tip couples to other ordinary modes NOT charge neutral:

  $R_A < 1$ on average $\rightarrow \frac{dI}{dV}$ decreases.

Law, Lee, Ng 2009

Flensberg 2010
In experiments so far, the measured zero bias peak height is around

\[
\frac{dI}{dV} < 0.1 \frac{e^2}{h}
\]

Probably due to normal modes coupling & finite temperature effect.

Fe atom chains-superconductor system, Yazdani group, Science 2014

TI-SC thin film experiment, Jia group, 2016

Hybrid superconductor-nanowire system, Kouwenhoven group, Science 2012
Topological insulators and superconductors

Full pairing gap in the bulk, gapless Majorana edge and surface states

Chiral Majorana fermions

Chiral fermions

Chiral SC

QH

Helical SC

QSH

massless Majorana fermions

massless Dirac fermions

(Qi, Hughes, Raghu and Zhang, PRL, 2009)
Chiral topological superconductivity from QAHE

(Qi, Hughes and Zhang PHYSICAL REVIEW B 82, 184516 (2010))

As one sweeps the magnetic field, there is NECESSARILY an intermediate phase with chiral topological superconductivity!
Longitudinal Conductance of QAH-TSC junction

Electron states $a_{1,2}, b_{1,2}$ decomposes into Majoranas on the edge of TSC.

Scattering matrix (amplitude):

$$
\begin{pmatrix}
\hat{b}_{1;e} \\
\hat{b}_{1;e}^\dagger \\
\hat{b}_{2;e} \\
\hat{b}_{2;e}^\dagger
\end{pmatrix} = \frac{1}{2}
\begin{pmatrix}
-1 & -1 & 1 & -1 \\
-1 & -1 & -1 & 1 \\
1 & -1 & 1 & 1 \\
-1 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
\hat{a}_{1;e} \\
\hat{a}_{1;e}^\dagger \\
\hat{a}_{2;e} \\
\hat{a}_{2;e}^\dagger
\end{pmatrix}
$$

Chung, et.al. PRB 83, 100512(R) (2011)
Longitudinal Conductance of QAH-TSC junction

An electron incident from left (right) has quantum probabilities:

\[
\begin{align*}
\text{transmission} \ T &= \frac{1}{4} , \quad \text{Andreev transmission} \ T_A &= \frac{1}{4} , \\
\text{reflection} \ R &= \frac{1}{4} , \quad \text{Andreev reflection} \ R_A &= \frac{1}{4} .
\end{align*}
\]

The generalized Laudauer-Buttiker formula:

\[
\begin{align*}
I_1 &= \frac{e^2}{h} \left[ (1 - R + R_A)(V_1 - V_{SC}) - (T - T_A)(V_2 - V_{SC}) \right] \\
-I_2 &= \frac{e^2}{h} \left[ (1 - R + R_A)(V_2 - V_{SC}) - (T - T_A)(V_1 - V_{SC}) \right] \\
I &= \frac{I_1 + I_2}{2} = \frac{e^2}{2h} (V_1 - V_2)
\end{align*}
\]

- When \( I_{SC} = 0 \), one has \( I = I_1 = I_2 \), and \( \sigma_{12} = \frac{e^2}{2h} \).
- Grounding of TSC is not needed if the TSC is sufficiently large.
Realizing chiral Majorana and TSC with QAH+SC

Experimental proposal

- S-wave SC covered on top of the middle region QAH sample
- Out-of-plane magnetic field applied
- Grounding of SC is not necessary if SC is large

Wang, et.al. PRB 92, 064520 (2015)
Exchange field $\lambda$ can be tuned by magnetic field $H$ (hysteresis), giving a half plateau in $\sigma_{12}$.
Experimental result
(K Wang+SC Zhang+K Liu+J Xia groups, Science)

Sample: \((Cr_{0.12}Bi_{0.26}Sb_{0.62})_2Te_3\) grown on \(GaAs(111)B\) supstrate, covered by \(Nb\) superconductor.

He, et.al.
Science 357, 294 (2017)
Exclusion of other explanations

The half-conductance plateau can also occur if the middle region is a metal, so that the two QAH form a series circuit:

\[ \sigma_{12} = \frac{e^2}{2h} \]

In this case, however, there is no integer plateau of \( \sigma_{12} = \frac{e^2}{h} \).
Large magnetic field resistance

For magnetic field > critical field of the superconductor, the middle region becomes metallic.

The system then becomes series connection of two QAH samples,

\[ \sigma_{12} = \frac{e^2}{2h} \]

This is verified in the experiment.
Three-terminal conductance

The 3-terminal measurements can be used as a further verification.

• The multi-terminal Landauer-Buttiker formula:

\[ I_i = \frac{e^2}{h} \left[ (k_i - R^i + R^i_A)(V_i - V_{SC}) - \sum_{j \neq i} (T^{ji} - T^{ji}_A)(V_j - V_{SC}) \right] \]

For middle region superconducting:

\[ \sigma_{13} = \sigma_{23} \leq e^2/h \quad \text{in N=1 TSC phase,} \quad \sigma_{13} = \sigma_{23} \approx 0 \quad \text{otherwise} \]
Three-terminal measurement

The 3-terminal measurements is also performed in the experiment, with terminal 3 implemented on Nb superconductor.

Wang & Zhang groups, Science

Theoretical prediction

Experimental measurements
Temperature Dependence of Half Plateau

The Majorana edge fermion takes the form:

$$\psi_k = u_k a_k + v_k a_k^\dagger$$

The scattering matrix is

$$\begin{pmatrix}
    a_{1,k}^\dagger \\
    a_{1,-k}^\dagger \\
    a_{2,k} \\
    a_{2,-k}^\dagger
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
    r & r & t+1 & t-1 \\
    r & r & t-1 & t+1 \\
    t^*+1 & t^*-1 & -r^* & -r^* \\
    t^*-1 & t^*+1 & -r^* & -r^*
\end{pmatrix} \begin{pmatrix}
    b_{1,k}^\dagger \\
    b_{1,-k}^\dagger \\
    b_{2,k} \\
    b_{2,-k}^\dagger
\end{pmatrix}$$

- At zero energy, $u_k = v_k = 1/\sqrt{2}$, $r = 1$, $t = 0$, and $\psi_k$ is charge neutral.
- At energy $\epsilon$, $u_k \neq v_k$, $r = r(\epsilon)$ and $t = t(\epsilon) = c_1 \epsilon + c_2 \epsilon^2 + \cdots$.

The conductance at temperature $T$ is

$$\sigma_{12} = \frac{e^2}{h} \int d\epsilon \left( -\frac{df}{d\epsilon} \right) \frac{1 + t(\epsilon)}{2} \approx \frac{e^2}{2h} (1 + \alpha T^2)$$

Here $f(\epsilon)$ is the Fermi-Dirac distribution.
Supercurrent Contribution

The bulk supercurrent is described by a boson field $\theta(x)$. The only allowed coupling of Majorana edge fermions with $\theta(x)$ is

$$H_{\text{eff}} = \frac{1}{2g} \int_{M_{\text{sc}}} d^2x \left[ (\partial_t \theta)^2 + v_s^2 (\nabla \theta)^2 \right] - i v_F \sum_{i=1}^4 \left[ (\psi_i \psi_{i+1} n_i \cdot \nabla \theta)_{x_i} + \int_{\partial_i M_{\text{sc}}} d\ell \psi_i \partial_\ell \psi_i \right]$$

This leads to a correction to the conductance

$$\delta \sigma_{12} = \frac{e^2}{2h} \frac{g \hbar}{16\pi^2 v_s} \sum_{ij} \langle n_i \cdot \nabla \theta(x_i) n_j \cdot \nabla \theta(x_j) \rangle \propto \frac{1}{L^3},$$

$L$ is the size of TSC.

- For $L \sim 1\mu m$, $\delta \sigma_{12} \sim 10^{-6} e^2/h$. 

After 80 years of searching, chiral Majorana fermion has been discovered!


Jing Wang, Quan Zhou, Biao Lian and Shou-Cheng Zhang, "Chiral topological superconductor and half-integer conductance plateau from quantum anomalous Hall plateau transition", Physical Review B, 92, 064520 (2015).

Qing. Lin He, Lei Pan, Alexander. L Stern, Edward Burks, Xiaoyu Che, Gen Yin, Jing Wang, Biao Lian, Quan Zhou and Eun. Sang Choi, Koichi Murata, Xufeng Kou, Tianxiao Nie, Qiming Shao, Yabin Fan, Shou-Cheng Zhang, Kai Liu, Jing Xia and Kang L. Wang, "Chiral Majorana edge state in a quantum anomalous Hall insulator-superconductor structure", Science 357, 294 (2017)
Chiral Majorana Edge State & Majorana Zero mode

A single chiral Majorana edge state implies the presence of *Majorana zero mode* (MZM) in a $\pi$ flux superconducting vortex.

![Diagram of chiral Majorana edge state and Majorana zero mode](image)

**Momentum quantization:**

$$k = \frac{2\pi n + \pi + \Phi}{L}, \quad n \in \mathbb{Z}$$

$2\pi$ spin rotation

magnetic flux

$\epsilon(k)$

Majorana operators:

$$\gamma_k = \gamma_{-k}^\dagger, \quad \gamma_0 = \gamma_0^\dagger$$
TSC in the presence of disorder

The chiral TSC phase is robust against disorders, as described by the percolation theory in the D symmetry class.

Zhang group, arXiv:1709.05558
Critical behavior of half plateau

The percolation theory determines the critical behaviors of the $\sigma_{12}$ **half plateau & plateau transitions** due to chiral TSC.

Size $L$ & temperature $T$ dependence:

- $e^2/2h$ to $e^2/h$: transition of middle region from TSC to QAHI

$$\frac{d\sigma_{12}}{dB} \propto L^{1/\nu_D} \text{ or } T^{-p'/2\nu_D} , \nu_D \approx 1$$

- 0 to $e^2/2h$: transition of left & right region from NI to QAHI

$$\frac{d\sigma_{12}}{dB} \propto L^{1/\nu_A} \text{ or } T^{-p/2\nu_A} , \nu_A \approx 7/3$$

- deviation of half plateau:

$$\delta\sigma_{12} \propto T^2$$

[Diagram of s-wave SC and QAHI]
Nonabelian braiding of chiral Majorana fermion

We can split one qubit into two chiral Majorana fermions. Natural propagation leads to non-abelian braiding. (Zhang group, arXiv:1712.06156)
A gate voltage applied to the QAH edge states introduces an additional phase. Applying periodic boundary condition to the TSC region leads to Corbino geometry.
Coherent braiding of chiral Majorana fermion demonstrated via Corbino geometry
Theoretical prediction of quantum oscillation
Angel particle

- In Dan Brown’s book “Angels and Demons”, the weight of the positron matter is $M(e^+) \sim 5 \times 10^{-6} \text{ g}$ yet the energy released is comparable to tons of TNT!

- Angel particle: we discovered a perfect world, with only angels, no demons! Angel particles may finally make quantum computers possible, bringing great benefit to humanity.