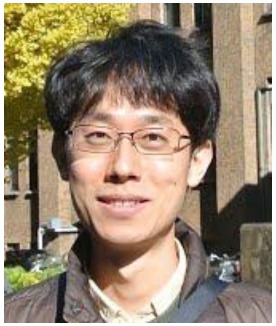
Break down of free-fermion classification of topological insulators and superconductors



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Christopher Mudry (Paul Scherrer Inst.)

Morimoto, AF, Mudry, Phys. Rev. B 92, 125104 (2015)

Plan of this talk

- Introduction
  - Classification of topological insulators and superconductors
    (gapped phases of free fermions)
    = classification of Dirac mass terms
- Interactions
  - Boundary fermions with dynamical Dirac mass terms
  - Topology of the space of dynamical Dirac masses
- Examples
  - 1D BDI, 3D DIII
  - Higher dimensions
  - 2D DIII + reflection, 3D AII + reflection

Generic discrete symmetries

- Time-reversal symmetry (TRS)

$$THT^{-1} = H$$

T : anti-unitary operator

$$\mathrm{TRS} = \begin{cases} 0 & \text{no TR invariance} \\ +1 & T^2 = +1 \\ -1 & T^2 = -1 \end{cases}$$

spin 0

spin 1/2

- Particle-hole symmetry (PHS)

BdG Hamiltonian for superconductors

$$CHC^{-1} = -H$$

C : anti-unitary operator

$$\mathrm{PHS} = \begin{cases} 0 & \text{no PH invariance} \\ +1 & C^2 = +1 \\ -1 & C^2 = -1 \end{cases}$$

Singlet SC

- Chiral symmetry (CS)

$$\Gamma H \Gamma^{-1} = -H$$

Γ : unitary operator

$$(\Gamma = TC)$$

$3 \times 3 + 1 = 10$

1: TRS = PHS = 0, CS = 0 or 1

Table of topological insulators/superconductors for d=1,2,3

	10 Symmetry Classes	TRS	PHS	CS	d=1	d=2	d=3
Standard (Wigner-Dyson)	A (unitary)	0	0	0	--	Z	--
	AI (orthogonal)	+1	0	0	--	--	--
	AII (symplectic)	-1	0	0	--	Z_2	Z_2
Chiral	AIII (chiral unitary)	0	0	1	Z	--	Z
	BDI (chiral orthogonal)	+1	+1	1	Z	--	--
	CII (chiral symplectic)	-1	-1	1	Z	--	Z_2
BdG	D (p-wave SC)	0	+1	0	Z_2	Z	--
	C (d-wave SC)	0	-1	0	--	Z	--
	DIII (p-wave TRS SC)	-1	+1	1	Z_2	Z_2	Z
	CI (d-wave TRS SC)	+1	-1	1	--	--	Z

Altland & Zirnbauer, PRB (1997)

Schnyder, Ryu, AF, and Ludwig, PRB (2008)

Class \ d	0	1	2	3	4	5	6	7	8	$\cdots$
complex case:										
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
real case:										
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

period $d = 2$

period $d = 8$

A. Kitaev, AIP Conf. Proc. 1134, 22 (2009); arXiv:0901.2686

Ryu, Schnyder, AF, Ludwig, NJP 12, 065010 (2010)

With interactions

Class	T	C	Γ_5	V_d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
A	0	0	0	C_{0+d}	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	C_{1+d}	$\mathbb{Z}_4$	0	$\mathbb{Z}_8$	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_{32}$	0
AI	+1	0	0	R_{0-d}	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	+1	+1	1	R_{1-d}	$\mathbb{Z}_8$	0	0	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	+1	0	R_{2-d}	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	−1	+1	1	R_{3-d}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0	$\mathbb{Z}_{32}$	0
AII	−1	0	0	R_{4-d}	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	−1	−1	1	R_{5-d}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0
C	0	−1	0	R_{6-d}	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	+1	−1	1	R_{7-d}	0	0	$\mathbb{Z}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{32}$	0

Reproduces:

1D BDI ($\mathbb{Z} \to \mathbb{Z}_8$), Fidkowski & Kitaev (2010)

3D DIII ($\mathbb{Z} \to \mathbb{Z}_{16}$), Kitaev (201?), Fidkowski-Chen-Vishwanath (2013),
Metlitski-Kane-Fisher (2014), …

…

With interactions

Freed & Hopkins (2016) stable homotopy theory; bordism

Class	T	C	Γ_5	V_d	$d=1$	$d=2$	$d=3$
A	0	0	0	C_{0+d}	0	$\mathbb{Z}$	0
AIII	0	0	1	C_{1+d}	$\mathbb{Z}_4$	0	$\mathbb{Z}_8$
AI	$+1$	0	0	R_{0-d}	0	0	0
BDI	$+1$	$+1$	1	R_{1-d}	$\mathbb{Z}_8$	0	0
D	0	$+1$	0	R_{2-d}	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	-1	$+1$	1	R_{3-d}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$
AII	-1	0	0	R_{4-d}	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	-1	-1	1	R_{5-d}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
C	0	-1	0	R_{6-d}	0	$\mathbb{Z}$	0
CI	$+1$	-1	1	R_{7-d}	0	0	$\mathbb{Z}_4$

Class	$d=1$	$d=2$	$d=3$
A	0	$\mathbb{Z}\times\mathbb{Z}$	0
AIII	$\mathbb{Z}_4$	0	$\mathbb{Z}_8\times\mathbb{Z}_2$
AI	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
BDI	$\mathbb{Z}_8$	0	0
D	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$
AII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$
CII	$\mathbb{Z}_2$	0	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$
C	0	$\mathbb{Z}\times\mathbb{Z}$	0
CI	$\mathbb{Z}_2$	0	$\mathbb{Z}_4\times\mathbb{Z}_2$

Reproduces:

1D BDI ($\mathbb{Z}\to\mathbb{Z}_8$), Fidkowski & Kitaev (2010)

3D DIII ($\mathbb{Z}\to\mathbb{Z}_{16}$), Kitaev (201?), Fidkowski-Chen-Vishwanath (2013), Metlitski-Kane-Fisher (2014), …

However, we do not have sectors of bosonic SPTs .

e.g., 3D AII: $\mathbb{Z}_2^3$, 3D AIII: $\mathbb{Z}_8\times\mathbb{Z}_2$, 3D CII: $\mathbb{Z}_2^5$, Wang & Senthil, PRB (2014)

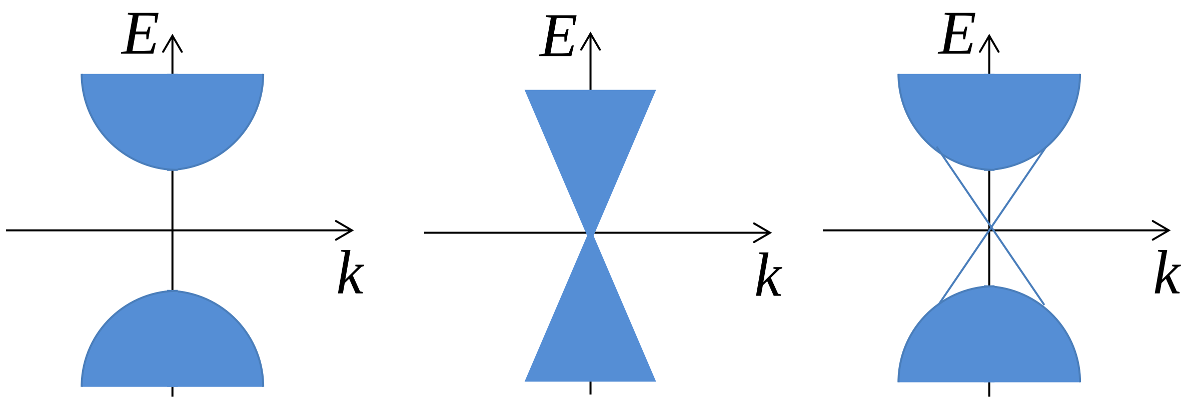
How to obtain the periodic table of TIs and TSCs

- Explicit construction of topological invariants
- Anderson delocalization of boundary fermions
- Quantum anomaly in the boundary theory
- Classification of Dirac masses in the bulk Hamiltonian
- K-theory
- …

Classification of Dirac mass terms

A. Kitaev (2009); T. Morimoto and AF, Phys. Rev. B 88, 125129 (2013)

Kitaev's method made simple!



Dirac Hamiltonian

$$H = \sum_{\mu=1}^{d} k_\mu \gamma_\mu + m\gamma_0$$

gamma matrices

$$\{\gamma_a, \gamma_b\} = 2\delta_{a,b}$$

minimal representative models for TIs and TSCs

effective theory for topological phase transitions (closing of a band gap)

classification of TIs and TSCs ⟺ classification of Dirac mass $m\gamma_0$

Massive Dirac Hamiltonian for TIs

$$H = \sum_{\mu=1}^{d} k_\mu \gamma_\mu + m\gamma_0 \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu,\nu}$$

Example

$d = 2$ class A (IQHE)

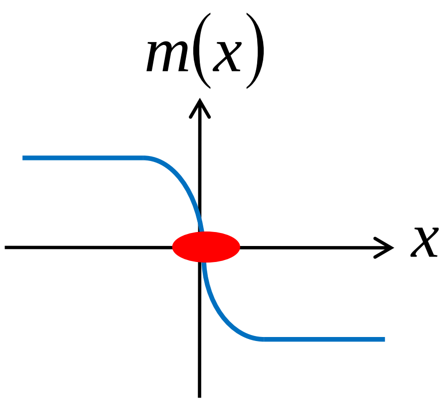
$$H = k_x\sigma_x + k_y\sigma_y + m\sigma_z \qquad \sigma_{yx} = \frac{1}{2}\mathrm{sgn}(m)$$

$m < 0$ $\qquad$ $m > 0$

Domain wall fermion = edge state

$$H = -iv(\partial_x\sigma_x - i\partial_y\sigma_y) + m(x)\sigma_z$$

$$E = -vk \qquad \psi(x,y) = \exp\left[iky - \frac{1}{v}\sigma_y\int_0^x m(x')dx'\right]\begin{pmatrix} 1 \\ -i \end{pmatrix}$$



Set of possible mass terms: classifying space

Example: $d = 2$ class A (IQHE)

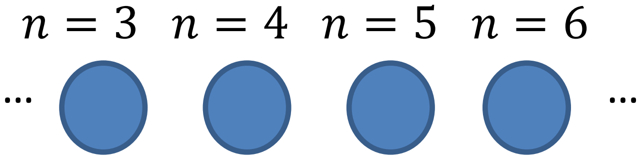
$$H = k_x \underbrace{\sigma_x \otimes 1_N}_{\gamma_1} + k_y \underbrace{\sigma_y \otimes 1_N}_{\gamma_2} + \gamma_0 \qquad \{\gamma_a, \gamma_b\} = 2\delta_{a,b}$$

$$\gamma_0 = \sigma_z \otimes A \qquad A = U \begin{pmatrix} 1_n & 0 \\ 0 & -1_m \end{pmatrix} U^\dagger \qquad (N = n + m)$$

$$\gamma_0 \Longleftrightarrow U \in U(n+m)/[U(n)\times U(m)]$$

Classifying space C_0

$$\pi_0\left[\oplus_n U(N)/U(N-n)\times U(n)\right] = \mathbb{Z}$$



There are topologically distinct gapped phases labelled by an integer index.

The parameter n corresponds to the Chern number.

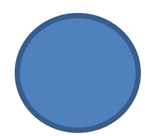
Example: $d=1$ class A (no symmetry constraint)

$$H = k_x \sigma_z \otimes 1_N + \gamma_0$$

$$\gamma_0 = \begin{pmatrix} 0 & U \\ U^\dagger & 0 \end{pmatrix} \qquad U \in U(N) \qquad \text{Classifying space } C_1$$

$$\pi_0\left(U(N)\right) = 0$$

There is only a single gapped phase.



Example: $d = 3$ class AII (time-reversal symmetry $T = i\sigma_y K$)

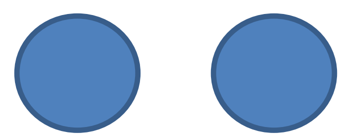
$$H = \left(k_x\sigma_x + k_y\sigma_y + k_z\sigma_z\right) \otimes \tau_z \otimes 1_N + \gamma_0$$

$$\gamma_0 = \sigma_0 \otimes \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix}_\tau$$

$$\gamma_0^2 = 1_{4N} \Longrightarrow X \in O(N)$$ Classifying space R_1

$$\pi_0\left(O\left(N\right)\right) = \mathbb{Z}_2$$

There are two gapped phases.



Sets of symmetry-allowed Dirac masses (classifying spaces V)

Class	T	C	Γ	Extension	V_d	$\pi_0(V_{d=0})$	$\pi_0(V_{d=1})$	$\pi_0(V_{d=2})$	$\pi_0(V_{d=3})$
A	0	0	0	$Cl_d \to Cl_{d+1}$	C_{0+d}	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	$Cl_{d+1} \to Cl_{d+2}$	C_{1+d}	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	$Cl_{0,d+2} \to Cl_{1,d+2}$	R_{0-d}	$\mathbb{Z}$	0	0	0
BDI	+1	+1	1	$Cl_{d+1,2} \to Cl_{d+1,3}$	R_{1-d}	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
D	0	+1	0	$Cl_{d,2} \to Cl_{d,3}$	R_{2-d}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	−1	+1	1	$Cl_{d,3} \to Cl_{d,4}$	R_{3-d}	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	−1	0	0	$Cl_{2,d} \to Cl_{3,d}$	R_{4-d}	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	−1	−1	1	$Cl_{d+3,0} \to Cl_{d+3,1}$	R_{5-d}	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	−1	0	$Cl_{d+2,0} \to Cl_{d+2,1}$	R_{6-d}	0	0	$\mathbb{Z}$	0
CI	+1	−1	1	$Cl_{d+2,1} \to Cl_{d+2,2}$	R_{7-d}	0	0	0	$\mathbb{Z}$

$C_{q+2} = C_q \quad R_{q+8} = R_q$

$\pi_0(V)$ counts the # of path-connected parts in the set V

$\pi_0(V) = 0$ trivial insulators

$\pi_0(V) = \mathbb{Z}, \mathbb{Z}_2$ topologically nontrivial insulators

Label	Classifying space V
C_0	$\cup_{n=0}^{N}\{\mathrm{U}(N)/[\mathrm{U}(n)\times\mathrm{U}(N-n)]\}$
C_1	$\mathrm{U}(N)$
R_0	$\cup_{n=0}^{N}\{\mathrm{O}(N)/[\mathrm{O}(n)\times\mathrm{O}(N-n)]\}$
R_1	$\mathrm{O}(N)$
R_2	$\mathrm{O}(2N)/\mathrm{U}(N)$
R_3	$\mathrm{U}(2N)/\mathrm{Sp}(N)$
R_4	$\cup_{n=0}^{N}\{\mathrm{Sp}(N)/[\mathrm{Sp}(n)\times\mathrm{Sp}(N-n)]\}$
R_5	$\mathrm{Sp}(N)$
R_6	$\mathrm{Sp}(N)/\mathrm{U}(N)$
R_7	$\mathrm{U}(N)/\mathrm{O}(N)$

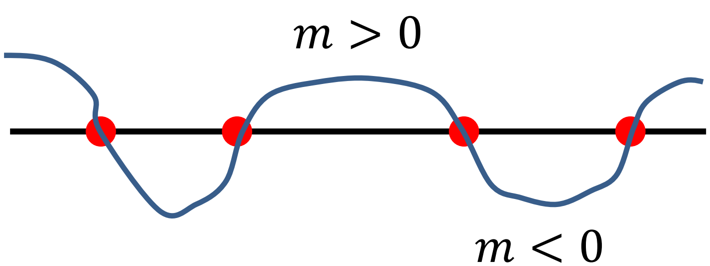
$\pi_n(V)$

Label	Classifying space V	$\pi_0(V)$	$\pi_1(V)$	$\pi_2(V)$	$\pi_3(V)$	$\pi_4(V)$
C_0	$\cup_{n=0}^{N}\{\mathrm{U}(N)/[\mathrm{U}(n)\times \mathrm{U}(N-n)]\}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
C_1	$\mathrm{U}(N)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
R_0	$\cup_{n=0}^{N}\{\mathrm{O}(N)/[\mathrm{O}(n)\times \mathrm{O}(N-n)]\}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
R_1	$\mathrm{O}(N)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
R_2	$\mathrm{O}(2N)/\mathrm{U}(N)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
R_3	$\mathrm{U}(2N)/\mathrm{Sp}(N)$	0	$\mathbb{Z}$	0	0	0
R_4	$\cup_{n=0}^{N}\{\mathrm{Sp}(N)/[\mathrm{Sp}(n)\times \mathrm{Sp}(N-n)]\}$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
R_5	$\mathrm{Sp}(N)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
R_6	$\mathrm{Sp}(N)/\mathrm{U}(N)$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
R_7	$\mathrm{U}(N)/\mathrm{O}(N)$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0

$\pi_n(V) \neq 0 \implies$ topological defects in the Dirac mass that can bind fermionic zero modes

$\pi_0(V) \neq 0$ Slowly varying random mass $\Rightarrow$ domains of constant mass

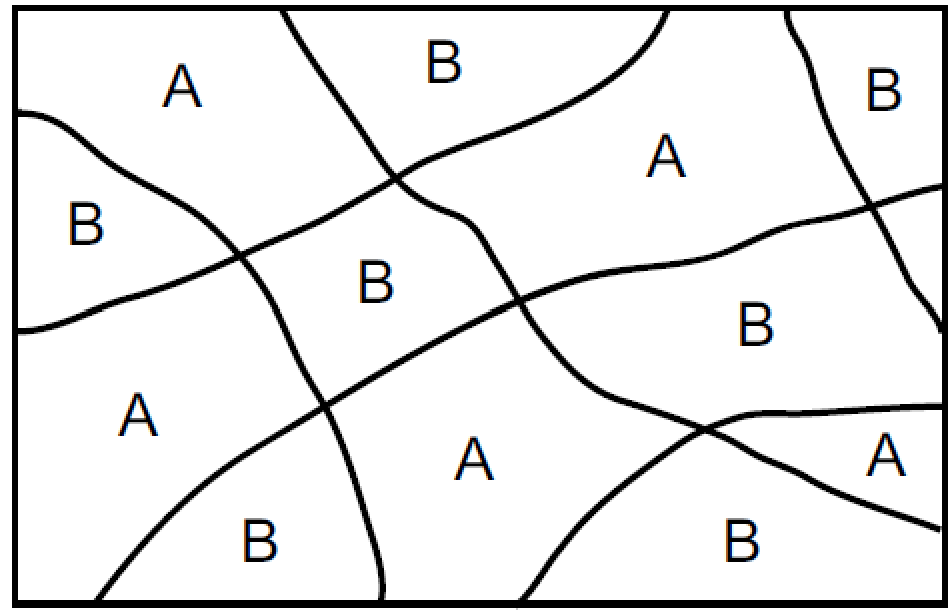
$d = 1$



Zeromodes at domain boundaries

Dyson singularity in DOS

$d = 2$



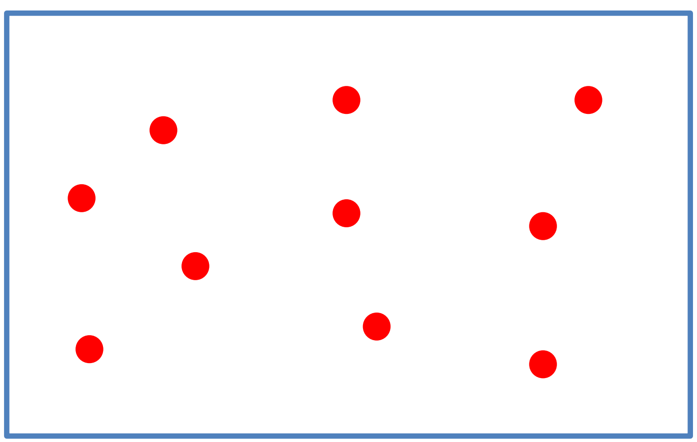
Domain walls have gapless edge states.
Quantum percolation of edge states leads to a critical point.

Cf. Chalker-Coddington model

$\pi_1(V) \neq 0$

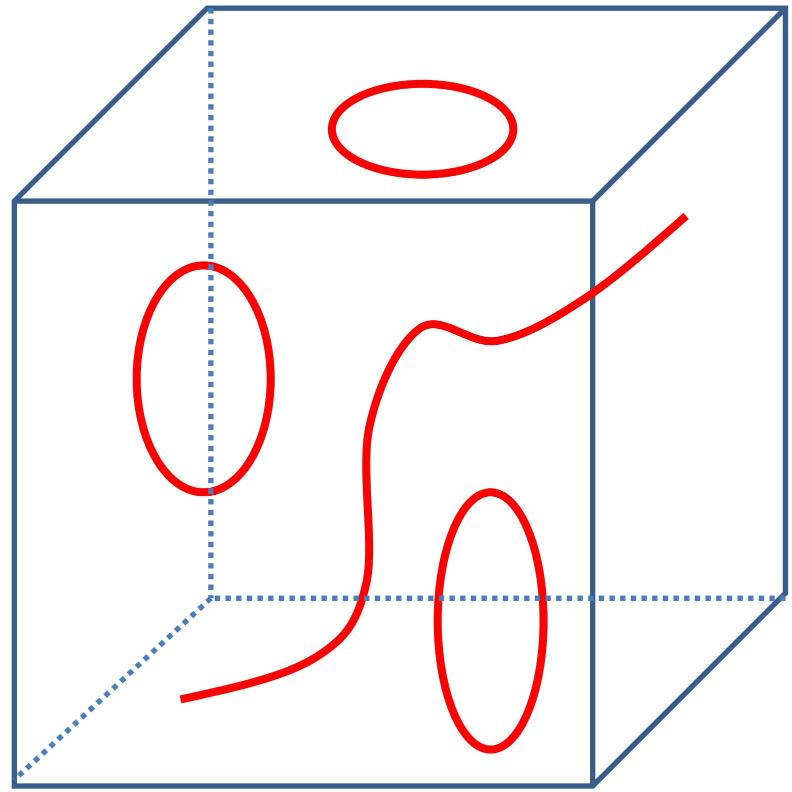
$d = 2$

point defects



$d = 3$

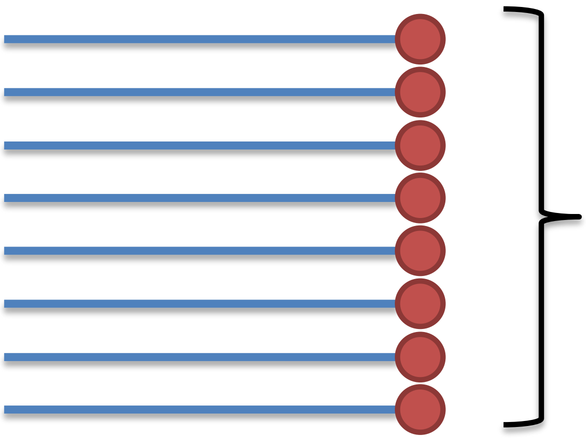
line defects



Interacting fermions

Reduction of non-interacting topological phases labeled by Z

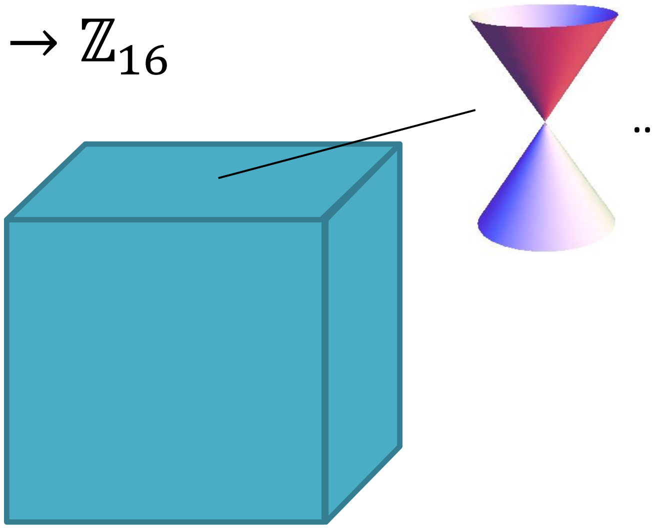
Time-reversal symmetric Majorana chain (1D class BDI)
$\mathbb{Z} \to \mathbb{Z}_8$



8 Majorana zero modes at the boundary can be gapped without breaking TRS.

Fidkowski and Kitaev, PRB (2010), PRB (2011)

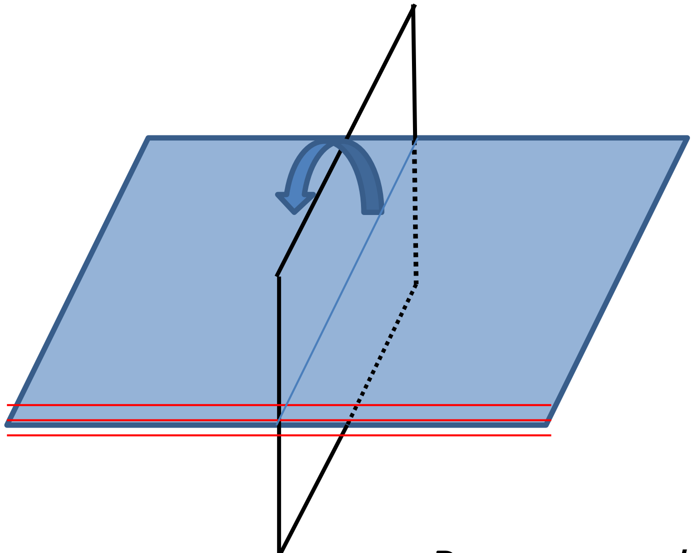
Time-reversal symmetric 3D topological SC (3D class DIII)
$\mathbb{Z} \to \mathbb{Z}_{16}$



$\nu \neq 16n$ flavors of Dirac surface fermions lead to nontrivial topological order with TRS.

Kitaev ; Fidkowski etal. PRX (2013), Metlitski, Kane & Fisher. (2014), ….

Time-reversal and reflection symmetric 2D superconductors (2D class DIII+R) $\mathbb{Z} \to \mathbb{Z}_8$



R: $x \to -x,\ i\sigma_x$

8 pairs of Majorana helical modes
can be gapped out by interactions
Yao and Ryu, PRB (2013); Qi, NJP (2013).

Aim:
Systematic study of the breakdown of the Z classification for any spatial dimension and all symmetry classes

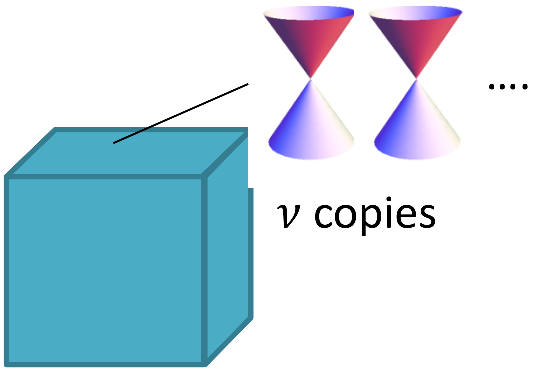
Q: Can boundary states be gapped out without breaking symmetries?

Stability analysis of boundary gapless states against interactions using the topology of the space of dynamical boundary Dirac masses

Kitaev, unpublished & talk @ UCLA (2015)

We only consider the contact interactions obtained from taking squares of the bilinears built from Dirac mass matrices $\left(\psi^{\dagger}\beta\psi\right)^{2}$

Our approach



ν copies of gapless boundary states

$$\mathcal{H}_0 = \sum_{j=1}^{d-1} (-i)\alpha_j \otimes 1_\nu \partial_j$$

Boundary massless Dirac fermions + quartic interactions

$$\mathcal{L}_{\mathrm{bd}} = \Psi^\dagger(\partial_\tau + \mathcal{H}_0)\Psi + \lambda \sum_{\{\beta\}} \left(\Psi^\dagger \beta_n \Psi\right)^2$$

$\{\beta_1, \beta_2, \dots, \beta_N\}$

marginal at $d-1=1$

irrelevant for $d-1>1$

We assume strong enough interactions when $d>2$ (but smaller than the bulk gap).

$\alpha \otimes 1$, β: mutually anti-commuting gamma matrices

$\alpha_j \otimes 1$ respect symmetries (such as TRS).

β_n are odd ($\beta_n \to -\beta_n$) under some symmetry transformation (T, R).

PHS is a sacred symmetry that we have to keep.

$$\mathcal{L}_{\mathrm{bd}} = \Psi^\dagger(\partial_\tau + \mathcal{H}_0)\Psi + \lambda \sum_{\{\beta\}} \left(\Psi^\dagger \beta_n \Psi\right)^2$$

Hubbard-Stratonovich transformtation

$$\mathcal{L}'_{\mathrm{bd}} = \Psi^\dagger \left(\partial_\tau + \mathcal{H}_{\mathrm{bd}}^{(\mathrm{dyn})}\right) \Psi + \frac{1}{\lambda} \sum_{n=1}^{N} \phi_n^2$$

$$\mathcal{H}_{\mathrm{bd}}^{(\mathrm{dyn})}(\tau, \boldsymbol{x}) = \mathcal{H}_0(\boldsymbol{x}) + \sum_{\{\beta\}} 2i\underline{\beta_n \phi_n(\tau, \boldsymbol{x})}$$

dynamical Dirac masses

$$\mathcal{H}_0 = \sum_{j=1}^{d-1} (-i)\alpha_j \otimes 1_\nu \partial_j$$

You and Xu, PRB (2014), Kitaev's talk @UCLA (2015)

Integrating out fermions

$$S_{\text{eff}}[\boldsymbol{\phi}] = -\text{Tr}\ \log\left[\partial_\tau + \sum_{j=1}^{d-1}(-i\partial_j)\alpha_j \otimes 1_\nu + \sum_{\{\beta\}} 2i\boldsymbol{\beta}\cdot\boldsymbol{\phi}\right] + \frac{1}{\lambda}\boldsymbol{\phi}^2$$

Saddle point approximation

+ including fluctuations about the direction in which ϕ freezes

Nonlinear sigma model

$$Z_{\text{bd}} \approx \int D[\phi]\delta(\boldsymbol{\phi}^2 - 1)e^{-S_{\text{QNLSM}}\boxed{-S_{\text{top}}}}$$

Abanov, Wiegmann Nucl. Phys. B (2000)

$$S_{\text{QNLSM}} = \frac{1}{g}\int d\tau \int d^{d-1}x\left(\partial_j\boldsymbol{\phi}\right)^2$$

S_{top} is a WZ term if $\pi_{d+1}(S^{N-1}) = \mathbb{Z}$.

$$\boldsymbol{\phi} \in S^{N(\nu)-1}$$

Target space of NLSM is a sphere generated by $N(\nu)$ anticommuting dynamical masses.

Topological obstructions to gapping

$\boldsymbol{\phi} \in S^{N(\nu)-1}$

The target space of NLSM is a sphere generated by $N(\nu)$ anticommuting dynamical masses.

Dirac fermions with dynamical masses in d-dimensional space time.

$\pi_0\left(S^{N(\nu)-1}\right) \neq 0$ domain wall

$\pi_1\left(S^{N(\nu)-1}\right) \neq 0$ vortex

$\vdots$

Topological defects in the dynamical masses bind fermion zero-energy states.

$\pi_d\left(S^{N(\nu)-1}\right) \neq 0$

$\pi_{d+1}\left(S^{N(\nu)-1}\right) \neq 0$ Wess-Zumino term

Boundary states cannot be gapped out without symmetry breaking or topological order.

Condition for the breakdown

$\pi_D\left(S^{N(\nu)-1}\right) = 0$ for $D = 0, \ldots, d+1$

$\nu_{\min}$: the minimum ν satisfying the condition $\qquad \mathbb{Z} \to \mathbb{Z}_{\nu_{\min}}$

Homotopy group of n-dimentional sphere

	S^1	S^2	S^3	S^4	S^5	S^6	S^7
π_1	Z	0	0	0	0	0	0
π_2	0	Z	0	0	0	0	0
π_3	0	Z	Z	0	0	0	0
π_4	0	Z_2	Z_2	Z	0	0	0
π_5	0	Z_2	Z_2	Z_2	Z	0	0
π_6	0	Z_{12}	Z_{12}	Z_2	Z_2	Z	0

Example 1: 1D class BDI

$\nu = 1$ Bulk: $\mathcal{H}^{(0)}(x) := -\mathrm{i}\partial_x \tau_3 + m(x)\,\tau_2.$

TRS $\mathcal{T} := \tau_1 \mathsf{K},$

PHS $\mathcal{C} := \tau_0 \mathsf{K}.$

Boundary: $\mathcal{H}^{(0)}_{\mathrm{bd}} = 0.$ $\mathcal{T}_{\mathrm{bd}} := \mathsf{K},\ \mathcal{C}_{\mathrm{bd}} := \mathsf{K}.$

ν copies

$$\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}\,\nu}(\tau) = \mathrm{i}M(\tau),$$

M : $\nu \times \nu$ Real anti-symmetric matrix

iM breaks TRS, but preserves PHS.

	Dynamical masses	Target space		Topological obstruction
ν=1	No mass term	-		-
ν=2	$iM = \pm\sigma_y$	pt.+pt.	$\pi_0 \neq 0$	Domain wall
ν=4	$iM = X_{21}, X_{02}, X_{23}$	S^2	$\pi_2 \neq 0$	WZ term
ν=8	$iM = X_{213}, X_{023}, X_{233}, X_{002}$	S^3 (S^6)		None $\mathbb{Z} \to \mathbb{Z}_8$

$$X_{ijk\cdots} = \sigma_i \otimes \sigma_j \otimes \sigma_k \otimes \cdots$$

Example 1: 1D class BDI

Space of dynamical Dirac masses at the boundary
= space Dirac masses for d=0 class D,
$R_2 = O(2N)/U(N)$

D	$\pi_D(R_2)$	ν	Topological obstruction
0	$\mathbb{Z}_2$	2	Domain wall
1	0		
2	$\mathbb{Z}$	4	WZ term
3	0		
4	0		
5	0		
6	$\mathbb{Z}$	8	None
7	$\mathbb{Z}_2$		

	Dynamical masses	Target space		Topological obstruction
ν=1	No mass term	-		-
ν=2	$iM = \pm\sigma_y$	pt.+pt.	$\pi_0 \neq 0$	Domain wall
ν=4	$iM = X_{21}, X_{02}, X_{23}$	S^2	$\pi_2 \neq 0$	WZ term
ν=8	$iM = X_{213}, X_{023}, X_{233}, X_{002}$	S^3 (S^6)		None $\mathbb{Z} \to \mathbb{Z}_8$

$$X_{ijk\cdots} = \sigma_i \otimes \sigma_j \otimes \sigma_k \otimes \cdots$$

Example 2: 3D class DIII

$X_{jk} = \sigma_j \otimes \sigma_k$

$\nu = 1$

$T = i\, X_{20} \mathrm{K}$
$C = \ X_{01} \mathrm{K}$

Bulk: $\mathcal{H}^{(0)}(\boldsymbol{x}) = -i\partial_1 X_{31} - i\partial_2 X_{02} - i\partial_3 X_{11} + m(\boldsymbol{x}) X_{03}$

Boundary: $\mathcal{H}^{(0)}_{\mathrm{bd}}(x,z) = -i\partial_x \sigma_3 - i\partial_z \sigma_1$

$T_{\mathrm{bd}} = i\sigma_2 \otimes 1_\nu \mathrm{K}$
$C_{\mathrm{bd}} = \ \sigma_0 \otimes 1_\nu \mathrm{K}$

ν copies

Dynamical mass: $\sigma_2 \otimes \underbrace{M(\tau, x, z)}_{\nu \times \nu \text{ real symmetric matrix}}$

breaks T, but preserves C.

<u>ν=1</u>: $M = \pm 1$ pt.+pt. $\pi_0 \neq 0$ domain wall

<u>ν=2:</u> $M = X_1, X_3$ $\pi_1(S^1) = \mathbb{Z}$ vortex

$X_{ijk\cdots} = \sigma_i \otimes \sigma_j \otimes \sigma_k \otimes \cdots$

<u>ν=4:</u> $M = X_{13}, X_{33}, X_{01}$ $\pi_2(S^2) = \mathbb{Z}$ monopole

<u>ν=8:</u> $M = X_{133}, X_{333}, X_{013}, X_{001}, X_{212}$ $\pi_4(S^4) = \mathbb{Z}$ WZ term $\mathbb{Z}_{16}$

Example 2: 3D class DIII

R_0: classifying space for 2D class D

D	$\pi_D(R_0)$	ν	Topological obstruction
0	$\mathbb{Z}$	1	Domain wall
1	$\mathbb{Z}_2$	2	Vortex line
2	$\mathbb{Z}_2$	4	Monopole
3	0		
4	$\mathbb{Z}$	8	WZ term
5	0		
6	0		
7	0		
8	$\mathbb{Z}$	16	None

<u>ν=1</u>: $M = \pm 1$ pt.+pt. $\pi_0 \neq 0$ domain wall

<u>ν=2:</u> $M = X_1, X_3$ $\pi_1(S^1) = \mathbb{Z}$ vortex

$$X_{ijk\cdots} = \sigma_i \otimes \sigma_j \otimes \sigma_k \otimes \cdots$$

<u>ν=4:</u> $M = X_{13}, X_{33}, X_{01}$ $\pi_2(S^2) = \mathbb{Z}$ monopole

<u>ν=8:</u> $M = X_{133}, X_{333}, X_{013}, X_{001}, X_{212}$ $\pi_4(S^4) = \mathbb{Z}$ WZ term $\mathbb{Z}_{16}$

Higher dimensions

- Z_2 in any dimension and any symmety class is stable.

  $^{\exists}D \leq d+1,\ \pi_D\left(S^{N(\nu)}\right) \neq 0$ for $\nu = 1$

- Z's in even dimensions are stable.

  Either (a) no dynamical mass exists, or (b) $\pi_1 \neq 0$.

- Z's in odd dimensions are unstable.

The reduction pattern is determined by the topology of the space of dynamical masses

	$d=8n+1$	$d=8n+3$	$d=8n+5$	$d=8n+7$
BDI	$\mathbb{Z} \to \mathbb{Z}_{2^{4n+3}}$	–	$\mathbb{Z} \to \mathbb{Z}_{2^{4n+4}}$	–
DIII	–	$\mathbb{Z} \to \mathbb{Z}_{2^{4n+4}}$	–	$\mathbb{Z} \to \mathbb{Z}_{2^{4n+5}}$
CII	$\mathbb{Z} \to \mathbb{Z}_{2^{4n+1}}$	–	$\mathbb{Z} \to \mathbb{Z}_{2^{4n+4}}$	–
CI	–	$\mathbb{Z} \to \mathbb{Z}_{2^{4n+2}}$	–	$\mathbb{Z} \to \mathbb{Z}_{2^{4n+5}}$

Class	T	C	Γ_5	V_d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
A	0	0	0	C_{0+d}	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	C_{1+d}	$\mathbb{Z}_4$	0	$\mathbb{Z}_8$	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_{32}$	0
AI	$+1$	0	0	R_{0-d}	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	$+1$	$+1$	1	R_{1-d}	$\mathbb{Z}_8$	0	0	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	$+1$	0	R_{2-d}	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	$+1$	1	R_{3-d}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0	$\mathbb{Z}_{32}$	0
AII	-1	0	0	R_{4-d}	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	R_{5-d}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0
C	0	-1	0	R_{6-d}	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	$+1$	-1	1	R_{7-d}	0	0	$\mathbb{Z}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{32}$	0

2D DIII + R (SC w/time-reversal & reflection)

$\nu = 1$ Bulk

$$\mathcal{H}^{(0)} = -i\partial_x X_{31} - i\partial_y X_{02} + m(x,y) X_{03} \qquad X_{ij} = \sigma_i \otimes \sigma_j$$

$$T = iX_{20}\mathrm{K}, \quad C = X_{01}\mathrm{K}, \quad R = iX_{20}$$

ν copies Boundary

$$\mathcal{H}_{\mathrm{bd}}^{(\mathrm{dyn})} = -i\partial_x \sigma_3 \otimes 1_\nu + M(\tau, x) \qquad M^* = -M \qquad C_{\mathrm{bd}} = \mathrm{K}$$

$$M = \begin{pmatrix} 0 & -iA \\ iA^T & 0 \end{pmatrix} \quad A \in O(\nu) = R_1$$

Classifying space for 1D class D

$\nu = 1$ $\pm\sigma_y$ $\pi_0 \neq 0$

$\nu = 2$ X_{21}, X_{23} $\pi_1(S^1) = \mathbb{Z}$

$\nu = 4$ $X_{210}, X_{230}, X_{102}, X_{222}$ $\pi_3(S^3) = \mathbb{Z}$

$\nu = 8$ $X_{2100}, X_{2310}, X_{2331}, X_{2333}, X_{1120}, \ldots$ $\pi_{1,2,3,4} = 0$ $\mathbb{Z}_8$

2D DIII + R (SC w/time-reversal & reflection)

D	$\pi_D(R_1)$	ν	Topological obstruction
0	$\mathbb{Z}_2$	1	Domain wall
1	$\mathbb{Z}_2$	2	Vortex
2	0		
3	$\mathbb{Z}$	4	WZ term
4	0		
5	0		
6	0		
7	$\mathbb{Z}$	8	None

R_1: classifying space for 1D class D

$\nu = 1 \quad \pm\sigma_y \quad \pi_0 \neq 0$

$\nu = 2 \quad X_{21},\ X_{23} \quad \pi_1(S^1) = \mathbb{Z}$

$\nu = 4 \quad X_{210},\ X_{230},\ X_{102},\ X_{222} \quad \pi_3(S^3) = \mathbb{Z}$

$\nu = 8 \quad X_{2100},\ X_{2310},\ X_{2331},\ X_{2333},\ X_{1120},\ \ldots \quad \pi_{1,2,3,4} = 0$ $\quad \mathbb{Z}_8$

3D topological crystalline insulators (e.g., SnTe)

(AII + reflection → $\mathbb{Z}$ classification)

Bulk:

$$\mathcal{H}^{(0)} = -i\partial_x X_{21} - i\partial_y X_{11} - i\partial_z X_{02} + m(\boldsymbol{x}) X_{03}$$

$$T = iX_{20}\mathrm{K}, \quad R_x = iX_{10}$$

Boundary: $\mathcal{H}_{\mathrm{bd}}^{(0)} = -i\partial_x \sigma_2 - i\partial_y \sigma_1$

We extend the surface Hamiltonian to the BdG form to allow for Cooper channels.

$$\mathcal{H}_{\mathrm{BdG}}^{(0)} = \mathcal{H}_{\mathrm{bd}}^{(0)} \otimes \left(-\mathcal{H}_{\mathrm{bd}}^{(0)}\right)^*$$

BdG: ν copies

$$\mathcal{H}_{\mathrm{bd}}^{(\mathrm{dyn})} = (-i\partial_x \sigma_2 \otimes \rho_3 - i\partial_y \sigma_1 \otimes \rho_0) \otimes 1_\nu + \gamma(x, y, z)$$

$$\mathcal{H}_{\text{bd}}^{(\text{dyn})} = \left(-i\partial_x\sigma_2\otimes\rho_3 - i\partial_y\sigma_1 \otimes \rho_0\right)\otimes 1_\nu + \gamma(x,y,z)$$

$$C_{\text{bd}} = \rho_1 K$$

$\nu = 1$ $\quad S^1 = \{c_1 X_{21} + c_2 X_{22} | c_1^2 + c_2^2 = 1\}$

$\nu = 2$ $\quad S^2 = \{c_1 X_{210} + c_2 X_{220} + c_3 X_{302} | c_1^2 + c_2^2 + c_3^2 = 1\}$

$\nu = 4$ $\quad S^4$

$\nu = 8$ $\quad \{X_{21000}, X_{22000}, X_{30200}, X_{30120}, X_{30312}, X_{30332}\}$

Space of masses (2D class D):

$$R_0 = \bigcup_{k=1}^{\nu} O(\nu)/\left[O(k)\times O(\nu-k)\right]$$

D	$\pi_D(R_0)$	ν	Topological obstruction
0	$\mathbb{Z}$		
1	$\mathbb{Z}_2$	1	Vortex
2	$\mathbb{Z}_2$	2	Monopole
3	0		
4	$\mathbb{Z}$	4	WZ term
5	0		
6	0		
7	0		
8	$\mathbb{Z}$	$\boxed{8}$	None

$\mathbb{Z} \to \mathbb{Z}_8$

In agreement with
Isobe & Fu, PRB (2015)

Summary

- Classification of TIs ⇔ classification of Dirac masses
- Reduction of the topological classification Z
  - Interactions can be represented in terms of dynamical Dirac masses (on the boundary)
  - Topology of the space of dynamical Dirac masses determines the pattern of reduction.

T. Morimoto, A.Furusaki, C.Mudry, Phys. Rev. B 92, 125104 (2015)

cf: R. Queiroz, E. Khalaf, A. Stern, Phys. Rev. Lett. 117, 206405 (2016)