Universal entropy of conformal critical points on a Klein bottle

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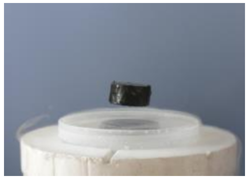
Outline

- Universal quantities for distinguishing quantum phases of matter
- 1+1d quantum systems at finite temperature: torus vs. Klein bottle partition functions
- Universal entropy of 1+1d non-chiral conformal field theories (CFTs) on a Klein bottle
- Summary and outlook

Conventional phases of matter

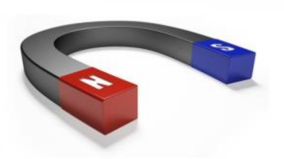
- Theoretical framework for characterizing conventional phases of matter:

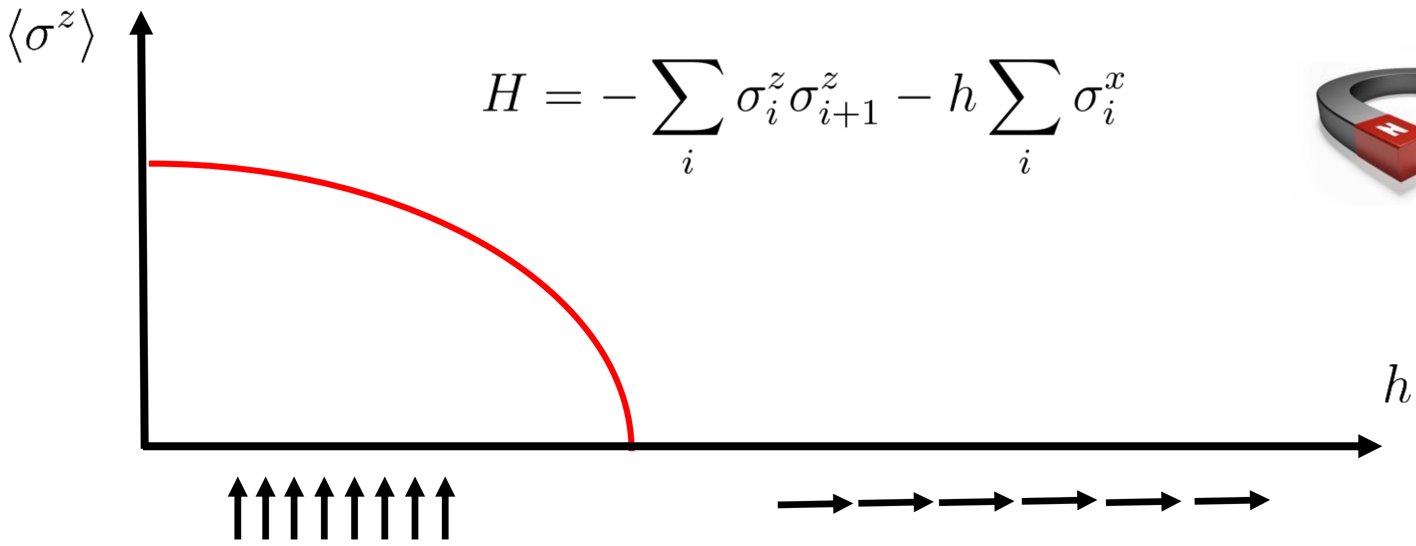
  Spontaneous symmetry breaking and associated local order parameters



Example: quantum Ising chain

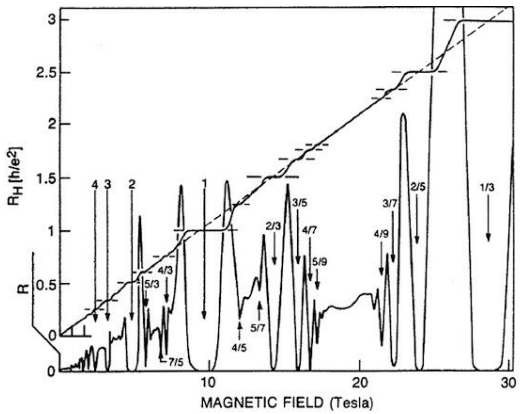
$$H = -\sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x$$



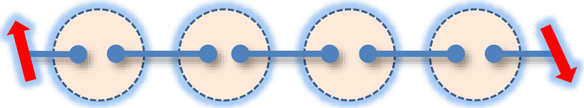


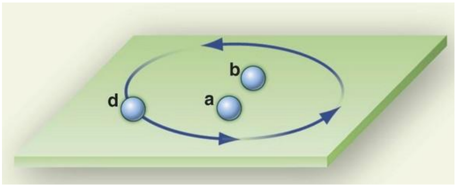
Topological phases of matter

- Topological phases: no symmetry breaking
  - Integer and fractional quantum Hall states
  - Spin-1 Haldane chains
  - …



- Exotic features:
  - Edge states
  - Ground-state degeneracy
  - Anyonic excitations
  - Fractional quantum numbers
  - …



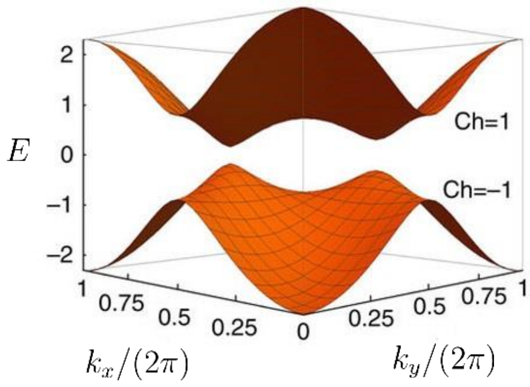


Universal data for distinguishing topological phases

- Topological invariant for free fermions, e.g. TKNN number for 2d Chern band

$$H = \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}} (\vec{d}_{\mathbf{k}} \cdot \vec{\sigma}) c_{\mathbf{k}}$$

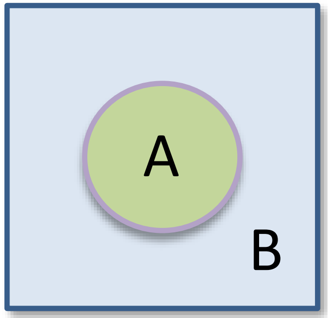
$$I_{\text{TKNN}} = \frac{1}{4\pi} \int \mathrm{d}k_x \mathrm{d}k_y \hat{\mathbf{d}} \cdot (\partial_{k_x} \hat{\mathbf{d}} \times \partial_{k_y} \hat{\mathbf{d}})$$



- Topological entanglement entropy in 2d:

$$S_A(L) = \alpha L - \ln \mathcal{D} + \cdots$$

$$\mathcal{D} = \sqrt{\sum_a d_a^2}$$: total quantum dimensions of anyons



$$S_A = -\mathrm{Tr}(\rho_A \ln \rho_A)$$

Thouless, Kohmoto, Nightingale, den Nijs, ’92

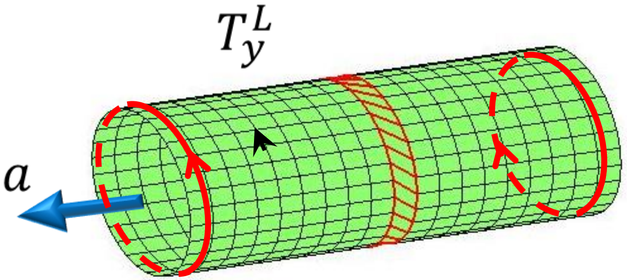
Kitaev & Preskill, ’06; Levin & Wen, ’06

Universal data for distinguishing topological phases

- Momentum polarization for 2d chiral topological states:

$$\langle G_a | T_L^y | G_a \rangle = \exp\left[i \frac{2\pi}{N_y} (h_a - \frac{c}{24}) - \alpha N_y \right]$$

topological spin　　chiral central charge



HHT, Y. Zhang & X.-L. Qi, ’13

Zaletel, Mong & Pollmann, ’13

This talk → **Universal data from thermal states**

Target: many-body problems for which *non-chiral* CFT is relevant

i) 1+1d quantum critical systems

ii) 2d classical statistical models at criticality

iii) 2+1d symmetry-protected topological states with gapless edge

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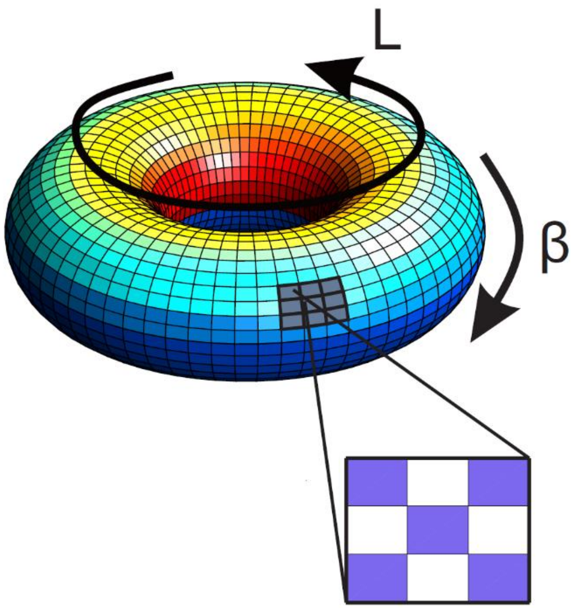
1d quantum model at finite T

$$H = -\sum_{i=1}^{L} \sigma_i^x \sigma_{i+1}^x - \sum_{i=1}^{L} \sigma_i^z$$



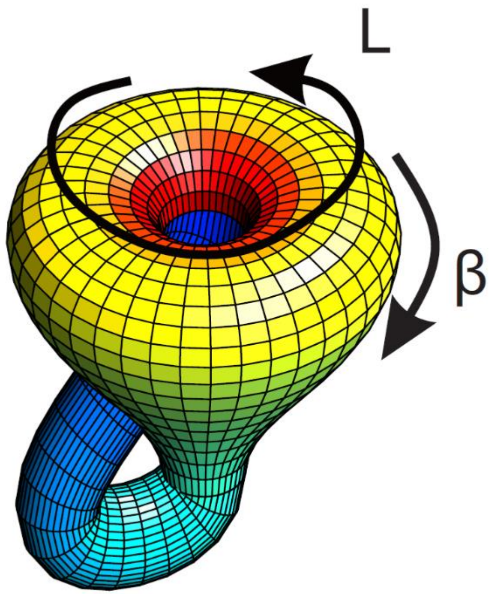
Partition function at $\beta = 1/T$:

$$\begin{aligned} Z^T &= \mathrm{Tr}(e^{-\beta H}) \\ &\simeq \mathrm{Tr}(e^{-\Delta\tau H_{\mathrm{even}}} e^{-\Delta\tau H_{\mathrm{odd}}} \cdots e^{-\Delta\tau H_{\mathrm{odd}}}) \end{aligned}$$

- The partition function of 1d quantum models with periodic boundary lives on a torus.

Klein bottle: a twist in imaginary time direction

$$H = -\sum_{i=1}^{L} \sigma_i^x \sigma_{i+1}^x - \sum_{i=1}^{L} \sigma_i^z$$



Klein bottle partition function:

$$Z^{\mathcal{K}} = \mathrm{Tr}(Pe^{-\beta H})$$

$$P|\sigma_1, \sigma_2, \ldots, \sigma_{L-1}, \sigma_L\rangle = |\sigma_L, \sigma_{L-1}, \ldots, \sigma_2, \sigma_1\rangle$$

- The partition function of 1d quantum models with periodic boundary and a spatial reflection twist (in imaginary time direction) lives on a Klein bottle.

1d critical Ising chain

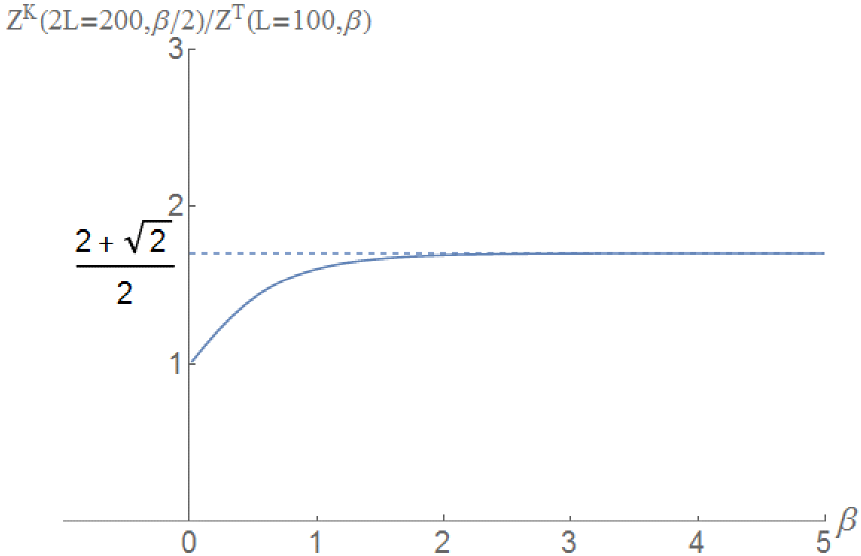
$$H = -\sum_{i=1}^{L} \sigma_i^x \sigma_{i+1}^x - \sum_{i=1}^{L} \sigma_i^z$$

Universal ratio for $L \gg \beta$:

(long chain, low temperature)

$$\frac{Z^{\mathcal{K}}(2L, \frac{\beta}{2})}{Z^{\mathcal{T}}(L, \beta)} = \frac{1+1+\sqrt{2}}{2}$$

$$\frac{d_I + d_\psi + d_\sigma}{\mathcal{D}}$$



Ising CFT: $d_I = d_\psi = 1, \ d_\sigma = \sqrt{2}$

$$\mathcal{D} = \sqrt{d_I^2 + d_\psi^2 + d_\sigma^2} = 2$$

HHT, arXiv:1707.05812

1d XY chain

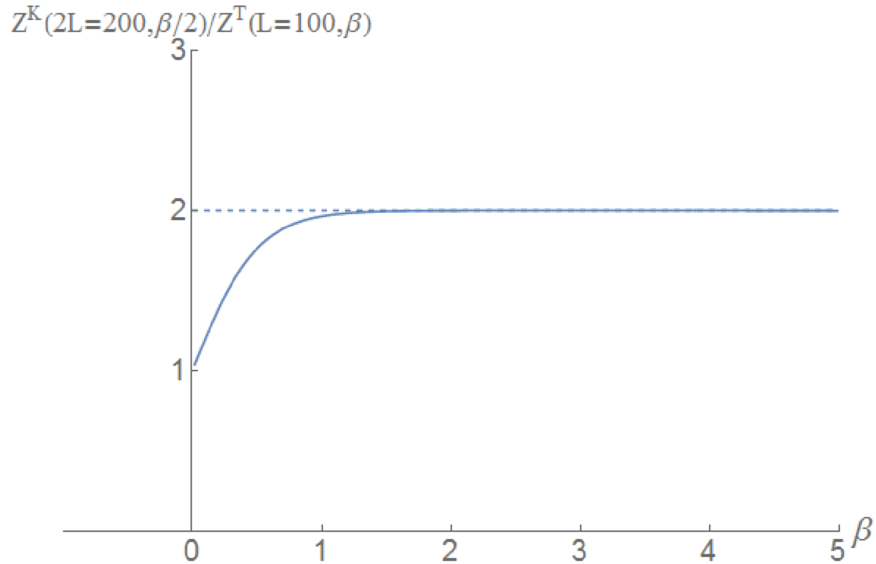
$$H = -\sum_{i=1}^{L} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y)$$

Universal ratio for $L \gg \beta$:

(long chain, low temperature)

$$\frac{Z^{\mathcal{K}}(2L, \frac{\beta}{2})}{Z^{\mathcal{T}}(L, \beta)} = \frac{1+1+1+1}{2}$$

$$\frac{d_I + d_s + d_{\bar{s}} + d_v}{\mathcal{D}}$$



U(1)$_4$ CFT: $d_I = d_s = d_{\bar{s}} = d_v = 1$

$$\mathcal{D} = \sqrt{d_I^2 + d_s^2 + d_{\bar{s}}^2 + d_v^2} = 2$$

HHT, arXiv:1707.05812

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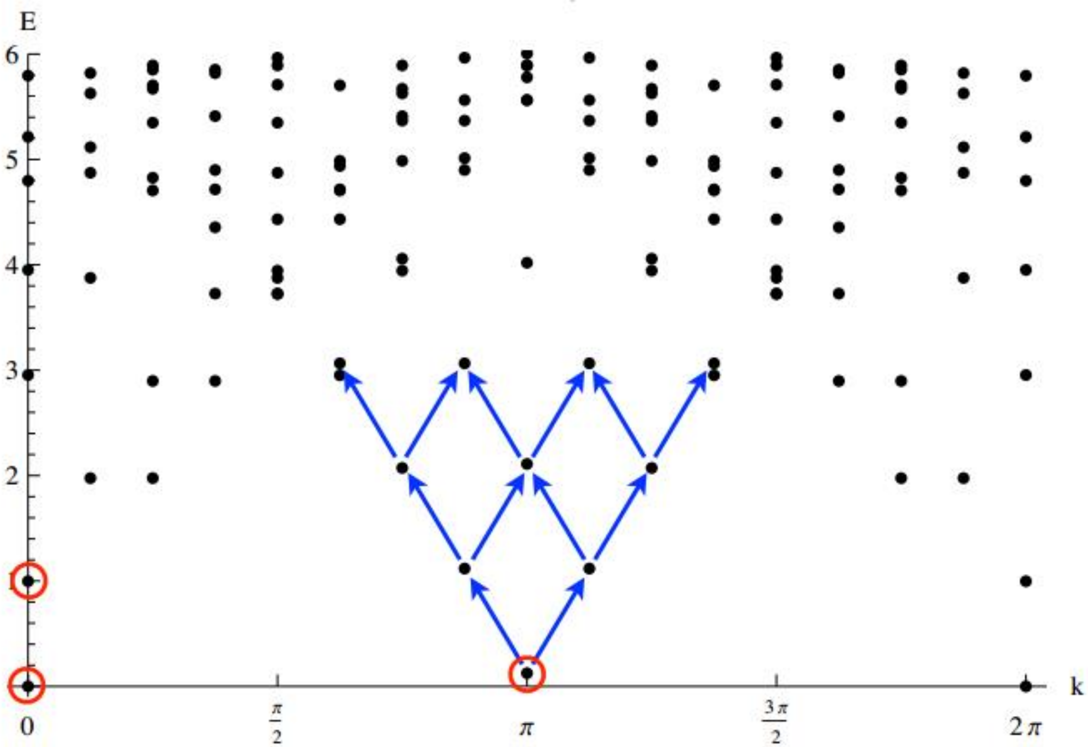
Energy spectrum of 1d critical Ising chain



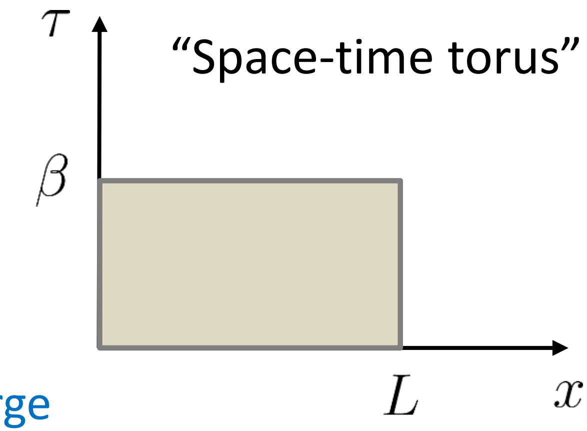
$$H = \sum_{i=1}^{L} \sigma_i^x \sigma_{i+1}^x - \sum_{i=1}^{L} \sigma_i^z$$

Plot by Eddy Ardonne

- Low-energy physics governed by non-chiral CFT: $H = P + \bar{P}$

Quick sketch of chiral CFT

- 1+1d chiral CFT: gapless right movers with linear dispersion



$$H = P$$

$$H = P = \frac{2\pi}{L}\left(L_0 - \frac{c}{24}\right)$$

central charge

Eigenvectors: $L_0|\alpha\rangle = h_\alpha|\alpha\rangle$ Vacuum: $L_0|0\rangle = 0$

conformal dimension

➢ Non-chiral CFTs, describing 1d quantum critical chains, are recovered by combining with a counter-propagating branch.

Virasoro algebra, primary field, and Verma module

Virasoro algebra: $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$

Primary state: $L_n|\alpha\rangle = 0,\ \forall\, n > 0$ ⇒ Primary field

$$|\alpha\rangle = \lim_{z\to 0} \mathcal{V}_\alpha(z)|0\rangle$$

Verma module:

$$\begin{array}{llllll}
h_\alpha & |\alpha\rangle & & & & \\
h_\alpha + 1 & L_{-1}|\alpha\rangle & & & & \\
h_\alpha + 2 & L_{-1}^2|\alpha\rangle & L_{-2}|\alpha\rangle & & & \\
\vdots & L_{-1}^3|\alpha\rangle & L_{-1}L_{-2}|\alpha\rangle & L_{-3}|\alpha\rangle & & \\
 & L_{-1}^4|\alpha\rangle & L_{-1}^2L_{-2}|\alpha\rangle & L_{-1}L_{-3}|\alpha\rangle & L_{-2}^2|\alpha\rangle & L_{-4}|\alpha\rangle \\
 & \vdots & & & &
\end{array}$$

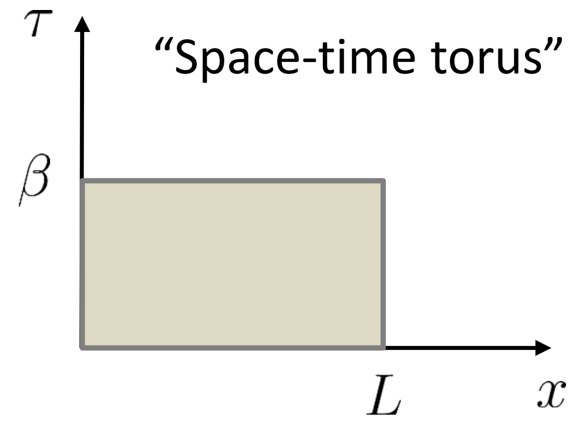
- Null vectors with vanishing norm should be removed, e.g., $L_{-1}|0\rangle = 0$

Chiral character of CFT

- Chiral character is like a partition function (restricted to each tower of states)

$$\begin{aligned}\chi_a(q) &= \mathrm{Tr}_a(e^{-\beta H}) \\ &= \mathrm{Tr}_a[e^{-2\pi\frac{\beta}{L}(L_0 - c/24)}] \\ &= \mathrm{Tr}_a(q^{L_0 - c/24})\end{aligned}$$

$$q = e^{-2\pi\frac{\beta}{L}}$$



Example: Ising CFT $\quad c = 1/2$

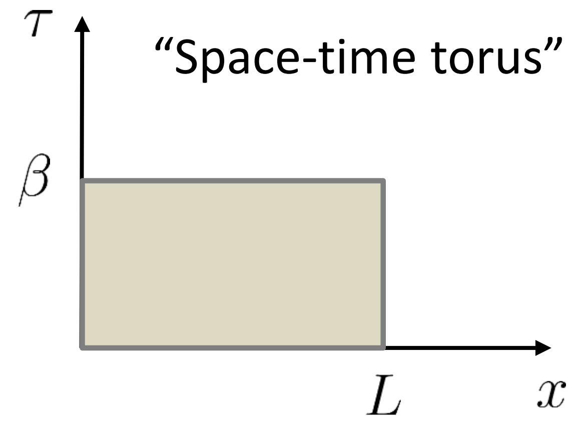
primaries $\quad I,\ \sigma,\ \psi \qquad h_I = 0,\ h_\sigma = \frac{1}{16},\ h_\psi = \frac{1}{2}$

Characters:

$$\begin{aligned}\chi_I(q) &= q^{-1/48}(1 + q^2 + q^3 + 2q^4 + 2q^5 + \ldots) \\ \chi_\sigma(q) &= q^{1/24}(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + \ldots) \\ \chi_\psi(q) &= q^{23/48}(1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + \ldots)\end{aligned}$$

Non-chiral CFT

- Non-chiral CFT: gapless left and right movers with linear dispersion



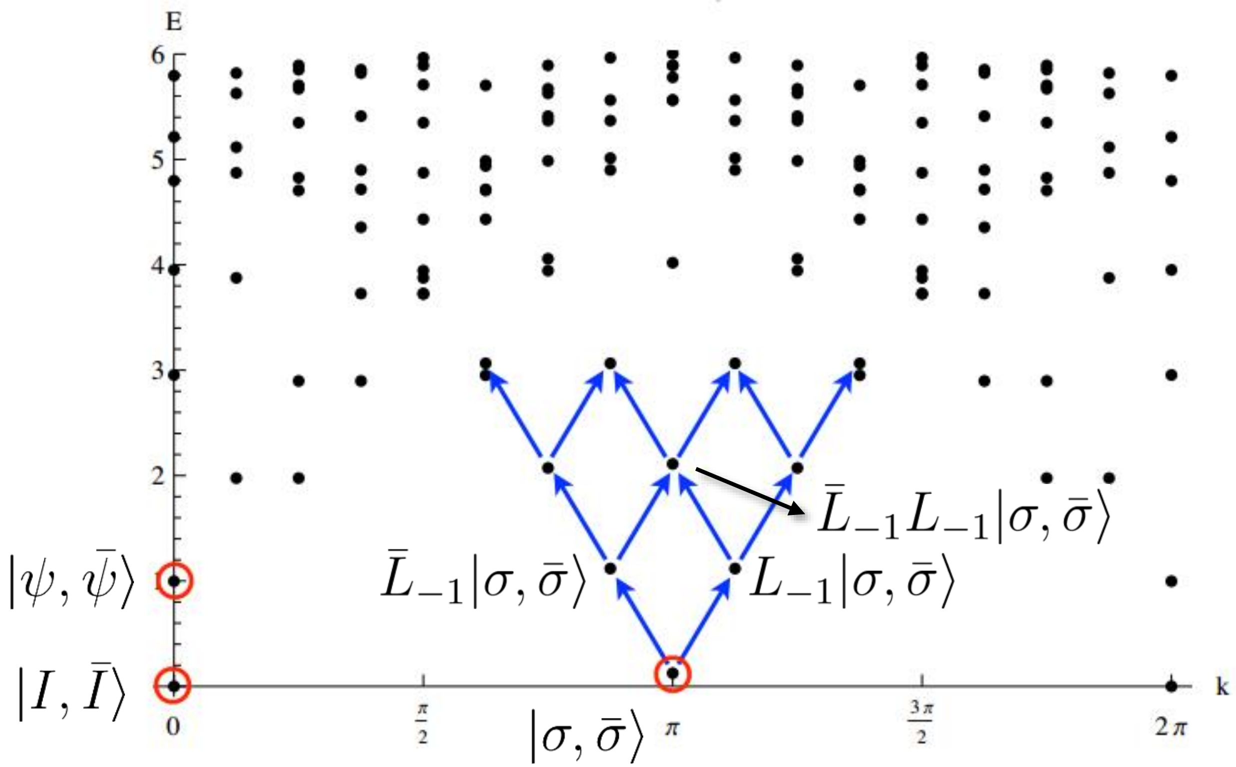
$$H = \frac{2\pi}{L}(L_0 - \frac{c}{24}) + \frac{2\pi}{L}(\bar{L}_0 - \frac{c}{24})$$
$$P = \frac{2\pi}{L}(L_0 - \bar{L}_0)$$

Eigenvectors: $L_0|\alpha,\bar{\gamma}\rangle = h_\alpha|\alpha,\bar{\gamma}\rangle \qquad \bar{L}_0|\alpha,\bar{\gamma}\rangle = h_\gamma|\alpha,\bar{\gamma}\rangle$

Partition function: $Z^{\mathcal{T}} = \mathrm{Tr}_{\mathcal{H}\times\bar{\mathcal{H}}}(q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24})$

➢ Non-chiral CFTs describe 1d critical chains and gapless edges of some 2d time-reversal invariant topological states.

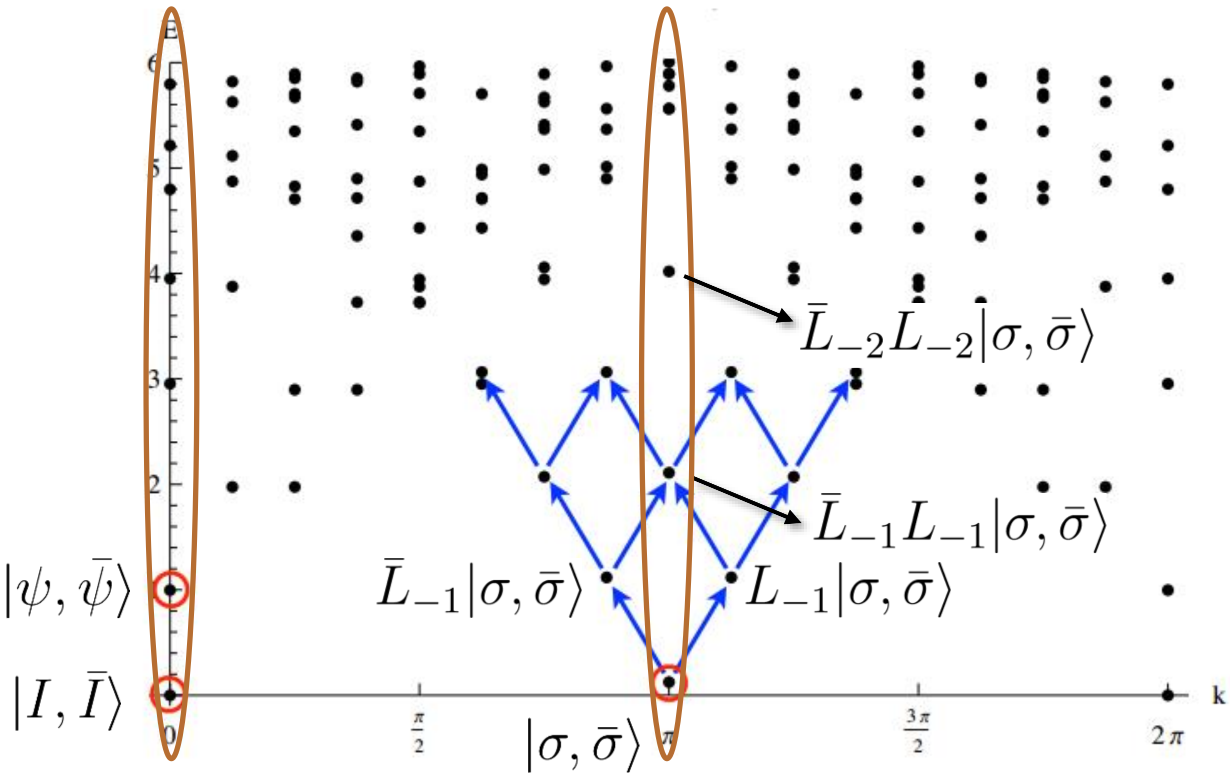
Torus partition function



Plot by Eddy Ardonne

$$Z^{\mathcal{T}} = \mathrm{Tr}(e^{-\beta H}) = \mathrm{Tr}_{\mathcal{H}\times\bar{\mathcal{H}}}(q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - c/24}) = \sum_{a=I,\psi,\sigma} \chi_a(q)\bar{\chi}_a(\bar{q})$$

Klein bottle partition function



$$P : k \to -k$$

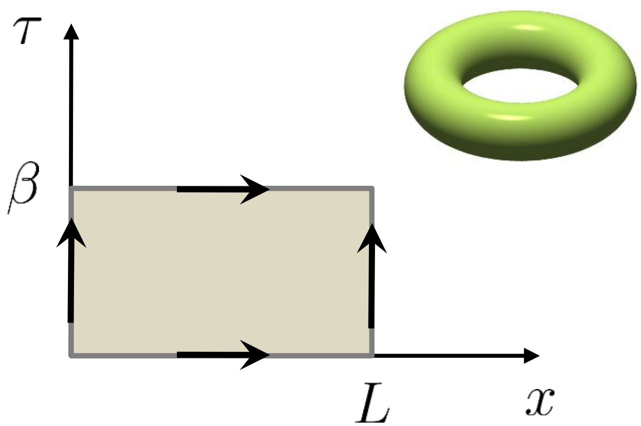
Plot by Eddy Ardonne

$$Z^{\mathcal{K}} = \mathrm{Tr}(Pe^{-\beta H}) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{sym}}}(q^{2L_0-\frac{c}{12}}) = \sum_{a=I,\psi,\sigma} \chi_a(q^2)$$

- Only left-right symmetric states contribute!

Torus vs. Klein bottle

Torus

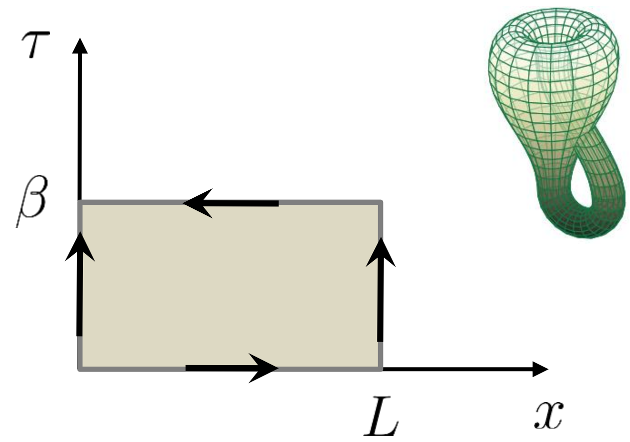


$$\ln Z^{\mathcal{T}}(L,\beta) \;\simeq\; -f_0\beta L + \frac{\pi c}{6\beta v}L$$

Affleck, '86

Blöte, Cardy, Nightingale, '86

Klein bottle



$$\ln Z^{\mathcal{K}}(L,\beta) \;\simeq\; -f_0\beta L + \frac{\pi c}{24\beta v}L + \ln g$$

HHT, arXiv:1707.05812

$$\boxed{\frac{Z^{\mathcal{K}}(2L,\frac{\beta}{2})}{Z^{\mathcal{T}}(L,\beta)} = g = \frac{1}{\mathcal{D}}\sum_a M_{a,a} d_a}$$

- "Boundary entropy" without boundary!
- This entropy can detect phase transitions.

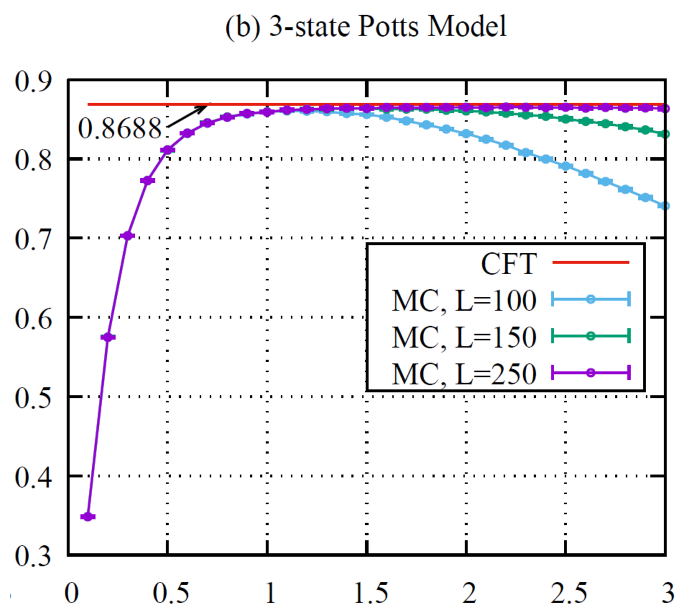
Numerical methods

- Monte Carlo: sampling the ratio of partition functions

Example: Z_3 quantum Potts chain

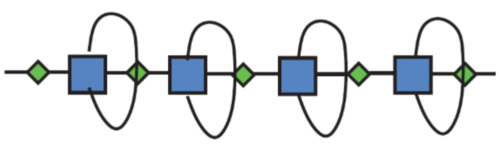
$$H = -\sum_{i=1}^{L}(\sigma_i^\dagger \sigma_{i+1} + \mathrm{h.c.}) - \sum_{i=1}^{L}(\tau_i + \tau_i^\dagger)$$

W. Tang, L. Chen, W. Li, HHT, L. Wang, '17

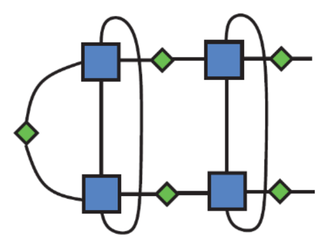


- Tensor network (Wei Li's group)

torus



Klein bottle



L. Chen, H.-X. Wang, L. Wang, W. Li, '17

Summary

- The 1+1d non-chiral CFTs have a universal entropy on a Klein bottle (relevant for 1+1d quantum critical systems, 2d classical statistical models at criticality, 2+1d SPT phases with gapless edges).
- The universal entropy can distinguish different CFTs and locate phase transitions.
- Outlook: perform the Klein twist calculation for 2+1d SPT phases.

Thank you for your attention!