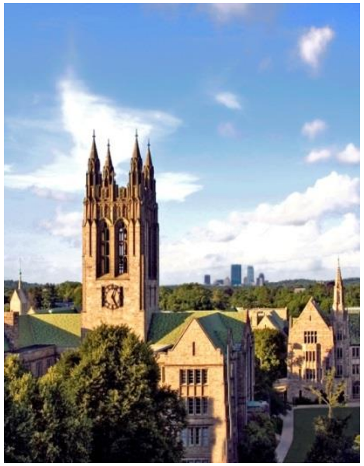
# Symmetry enforced SPT phases

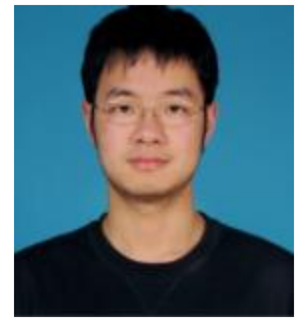
Ying Ran (Boston College)



KITS, July 2017

# Acknowledgement:

- **Collaborators for this project:**



  - Boston College Xu Yang Shenghan Jiang→Caltech
  - Harvard University Ashvin Vishwanath

- **Collaborators on an inspiring project (filling enforced quantum Hall):**
  - Ohio State Univ. Yuan-Ming Lu
  - Univ of Tokyo Masaki Oshikawa

References: arXiv:1611.07652, arXiv:1705.09298, arXiv:1705.05421

# Two types of symmetric topological phases

**SPT** --- symmetry protected topological phases: (Pollmann, Berg, Turner, Oshikawa, Chen, Liu, Gu, Wen, Wang, Senthil, Lu, Vishwanath, Zaletel, Cheng…)

Example: Haldane spin chain, integer quantum Hall states, topological insulators…

*Features*:

- no topological order
- anomalous edge states protected by symmetry

**SET** --- symmetry enriched topological phases: (Wen, Essin, Hermele, Mesaros,YR, Barkeshli, Chen, Wang, Senthil, Lu, Vishwanath, Zaletel, Watanabe, Cheng, Bonderson….)

Example: toric code, gapped quantum spin liquids, fractional quantum Hall states…

*Features*:

- topological order (anyon excitations in 2d)
- symmetry can be fractionalized (e.g. e/3 quasiparticle in Laughlin's state).

# Motivations

- Are there guiding principles to search for topological phases in strongly correlated quantum systems?

# Motivations: usual translation symmetry

- Hastings-Oshikawa-Lieb-Schultz-Mattis theorems (HOLSM) put strong constraints on symmetric quantum ground states (liquid phases)

(1) At a fractional filling, translation symmetry forbids a gapped short-range entangled ground state.

# Motivations: usual translation symmetry

- Hastings-Oshikawa-Lieb-Schultz-Mattis theorems (HOLSM) put strong constraints on symmetric quantum ground states (liquid phases)

(1) At a fractional filling, translation symmetry forbids a gapped short-range entangled ground state.

(2) With a nontrivial projective representation per unit cell, translation symmetry forbids a gapped short-range entangled ground state.

(Lieb, Schultz, Mattis, Oshikawa, Hastings, Watanabe, Po, Vishwanath, Zaletel…)

# Motivations: usual translation symmetry

- Hastings-Oshikawa-Lieb-Schultz-Mattis theorems (HOLSM) put strong constraints on symmetric quantum ground states (liquid phases)

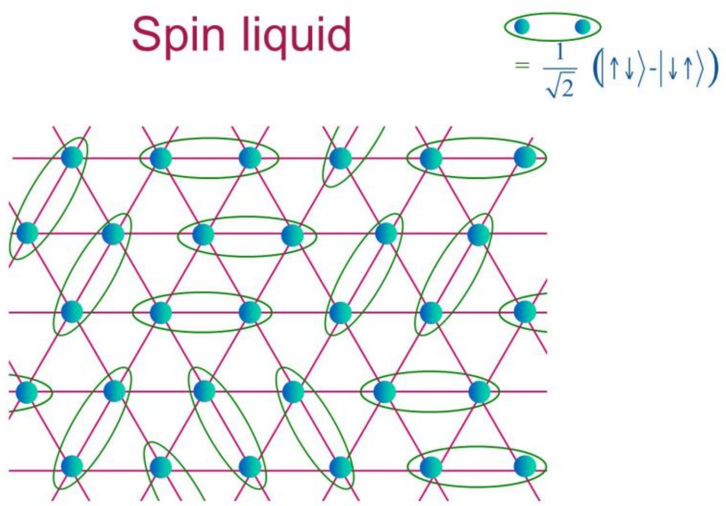
(1) At a fractional filling, translation symmetry forbids a gapped short-range entangled ground state.

(2) With a nontrivial projective representation per unit cell, translation symmetry forbids a gapped short-range entangled ground state.

(Lieb, Schultz, Mattis, Oshikawa, Hastings, Watanabe, Po, Vishwanath, Zaletel…)

In these cases, a gapped liquid phase must be topologically ordered.

HOLSM have been served as guiding principles to search for realizations of topologically ordered phases. (e.g., quantum spin liquids…)



# Motivations: Magnetic translation symmetry

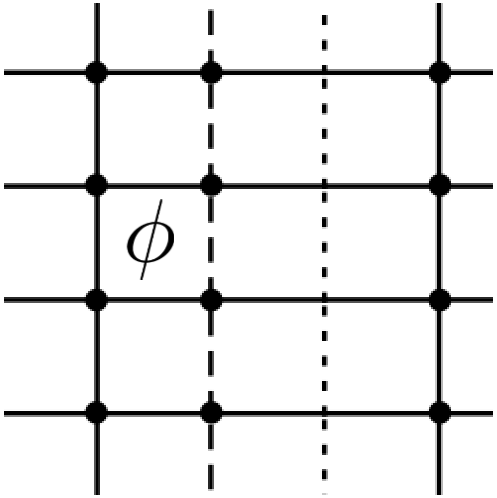
- What happens with magnetic translation symmetry?

e.g., Charge-conserving particles hopping on square lattice with flux per unit cell

$$T_x = e^{i\phi \sum_{\vec{r}} y n_{\vec{r}}} \cdot T_x^{orig.}, \quad T_y = T_y^{orig.}$$

$$T_x T_y T_x^{-1} T_y^{-1} = e^{i\phi \hat{N}}$$

$$\hat{N} = \sum_{\vec{r}} n_{\vec{r}}$$ A global U(1) rotation




$\phi = 2\pi/3$ Landau gauge

# Motivations: Magnetic translation symmetry

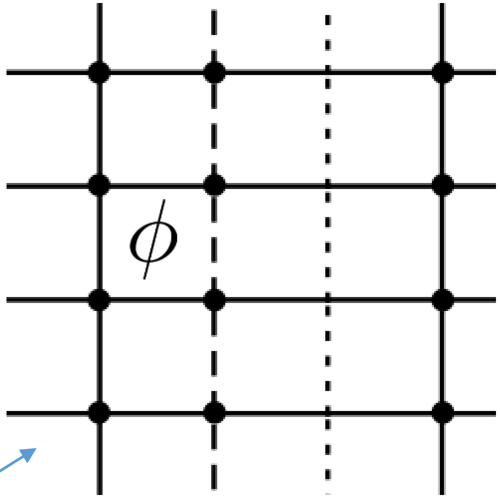
- What happens with magnetic translation symmetry?

e.g., Charge-conserving particles hopping on square lattice with flux per unit cell

$$T_x = e^{i\phi \sum_{\vec{r}} y n_{\vec{r}}} \cdot T_x^{orig.}, \quad T_y = T_y^{orig.}$$

$$T_x T_y T_x^{-1} T_y^{-1} = e^{i\phi \hat{N}}$$

$\hat{N} = \sum_{\vec{r}} n_{\vec{r}}$ A global U(1) rotation




Consider free fermions with weak lattice potential:
If $\nu = 1/3$ filling in the original unit cell, obviously one can have a gapped free fermion ground state:
an integer quantum Hall insulator from Landau level

Chern number =1

$\phi = 2\pi/3$ Landau gauge

# Motivations: Magnetic translation symmetry

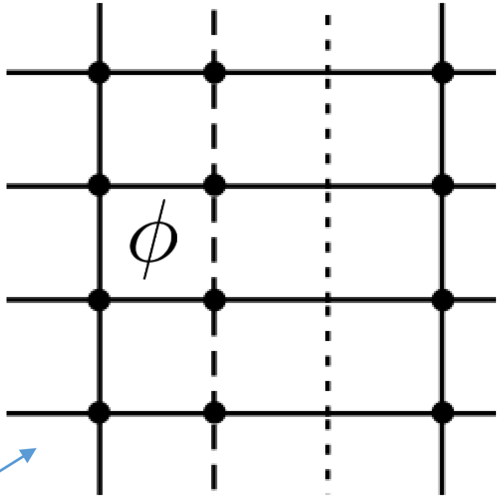
- What happens with magnetic translation symmetry?

e.g., Charge-conserving particles hopping on square lattice with flux per unit cell

$$T_x = e^{i\phi \sum_{\vec{r}} y n_{\vec{r}}} \cdot T_x^{orig.}, \quad T_y = T_y^{orig.}$$

$$T_x T_y T_x^{-1} T_y^{-1} = e^{i\phi \hat{N}}$$

$\hat{N} = \sum_{\vec{r}} n_{\vec{r}}$ A global U(1) rotation




$\phi = 2\pi/3$ Landau gauge

In the presence of arbitrary interaction, one can show:
If $\nu = 1/3$ filling in the original unit cell,
then unique ground state respecting magnetic translation dictates: Chern number $= 1 \ mod \ 3$

(Lu, Ran, Oshikawa, to appear)

# Motivations: Magnetic translation symmetry

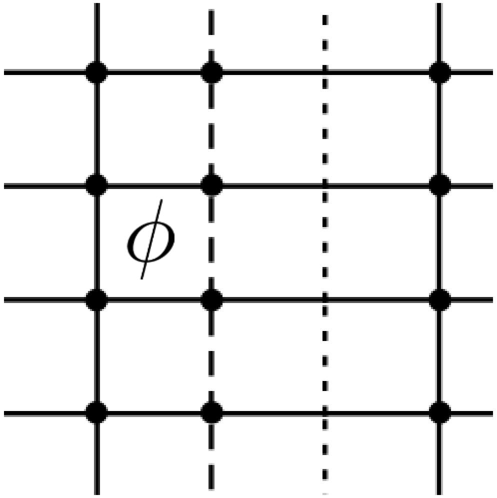
- What happens with magnetic translation symmetry?

e.g., Charge-conserving particles hopping on square lattice with flux per unit cell

$$T_x = e^{i\phi \sum_{\vec{r}} y n_{\vec{r}}} \cdot T_x^{orig.}, \quad T_y = T_y^{orig.}$$

$$T_x T_y T_x^{-1} T_y^{-1} = e^{i\phi \hat{N}}$$

$\hat{N} = \sum_{\vec{r}} n_{\vec{r}}$ A global U(1) rotation




$\phi = 2\pi/3$ Landau gauge

In the presence of arbitrary interaction, one can show:
If $\nu = 1/3$ filling in the original unit cell,
then unique ground state respecting magnetic translation dictates: Chern number $= 1 \; mod \; 3$

In this example, we learn that HOLSM + magnetic translation could lead to symmetry-enforced "integer" phases. Can one generalize this phenomena?

Wu, Ho and Lu, 2017 generalizes this for quantum spin Hall

# The question

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a \, {}^aU_b = \alpha(a,b)U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

$T_x, T_y$ are usual translation operations together with certain site-dependent local unitaries.

# The question

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\, {}^a U_b = \alpha(a,b) U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

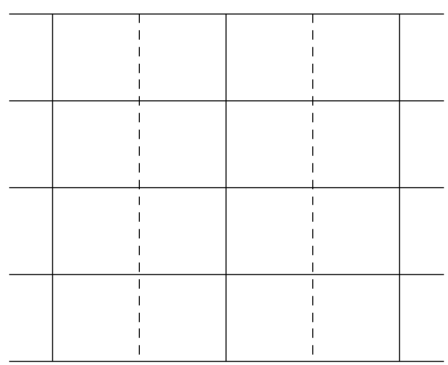
For example, fully frustrated Ising models on the square and honeycomb lattice satisfy this algebra with *g*=Ising:

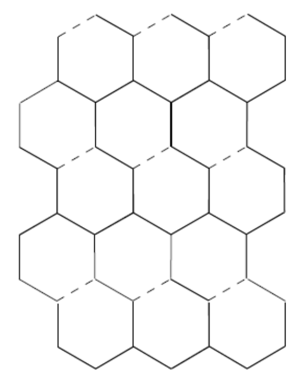
$$H = -\sum_{<IJ>} s_{IJ} \sigma_I^z \sigma_J^z$$

$s_{IJ} = +1$ on solid bond

$s_{IJ} = -1$ on dashed bond

Ising symm.: $\prod_I \sigma_I^x$





# The question

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\, {}^aU_b = \alpha(a,b)U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

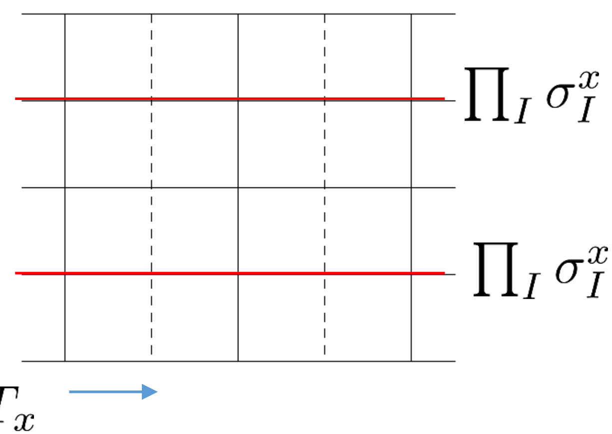
For example, fully frustrated Ising models on the square and honeycomb lattice satisfy this algebra with *g*=Ising:

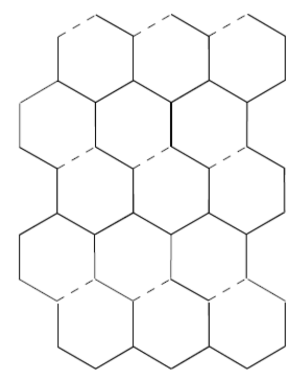
$$H = -\sum_{<IJ>} s_{IJ}\sigma^z_I \sigma^z_J$$

$s_{IJ} = +1$ on solid bond

$s_{IJ} = -1$ on dashed bond

Ising symm.: $\prod_I \sigma^x_I$



# The question

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\ {}^a U_b = \alpha(a,b) U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

We ask the following question:

Is a gapped SRE liquid (sym-SRE) phase possible? If the answer is yes, what kinds of sym-SRE phases are realizable?

# The question

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\, {}^aU_b = \alpha(a,b)U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

We ask the following question:

Is a gapped SRE liquid (sym-SRE) phase possible? If the answer is yes, what kinds of sym-SRE phases are realizable?

e.g.: G=SO(3)x Ising: spin-1/2 per unit cell, with Ising magnetic translation.

Based on rather complete understanding of bosonic sym-SRE phases, hopefully one can obtain systematic results.

# The question

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\ {}^aU_b = \alpha(a,b)U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

We ask the following question:

Is a gapped SRE liquid (sym-SRE) phase possible? If the answer is yes, what kinds of sym-SRE phases are realizable?

PLAN:

1. Main results: a theorem
2. A corollary: new HOLSM-type constraint
3. Sketch of the proof
4. Model realizations (A simple model realizing SPT phase)

# The question

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\ {}^aU_b = \alpha(a,b)U_{ab}$

and respecting magnetic translation symmetry:

$$T_xT_yT_x^{-1}T_y^{-1} = g \qquad g \in \text{ Center of } G$$

We ask the following question:

Is a gapped SRE liquid (sym-SRE) phase possible? If the answer is yes, what kinds of sym-SRE phases are realizable?

PLAN:

1. Main results: a theorem
2. A corollary: new HOLSM-type constraint
3. Sketch of the proof
4. Model realizations (A simple model realizing SPT phase)

# Main results

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a \, {}^aU_b = \alpha(a,b) U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

**Theorem**:

This system could have a sym-SRE phase if and only if there is a 3-cocycle $\omega_0(a,b,c) \in H^3(G, U(1))$ such that the slant product:

$\delta_g^{\omega_0}(a,b) \equiv \frac{\omega(g,a,b)\omega(a,b,g)}{\omega(a,g,b)}$ satisfies $\delta_g^{\omega_0} \simeq \alpha^{-1} \in H^2(G, U(1))$

When this condition is satisfied, the 3-cocyles of realizable sym-SRE phases form a coset: $\omega_0 \cdot \mathcal{A}_g$, where $\mathcal{A}_g$ is the kernel of the slant product. And all such realizable sym-SRE phases are nontrivial SPTs.

# Main results

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a \, {}^{a}U_b = \alpha(a,b) U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

**Theorem**:

This system could have a sym-SRE phase if and only if there is a 3-cocycle

$\omega_0(a,b,c) \in H^3(G, U(1))$ such that the slant product:

$\delta_g^{\omega_0}(a,b) \equiv \frac{\omega(g,a,b)\omega(a,b,g)}{\omega(a,g,b)}$ satisfies $\delta_g^{\omega_0} \simeq \alpha^{-1} \in H^2(G, U(1))$

Physical meaning: a sym-SRE phase is realizable if and only if its g-symmetry-defect carries the projective representation $\alpha$ .

Mathematically the slant product $\delta_g$ is computing the projective representation carried by the g-defect. (discussed by Zaletel 2013 without proof, we provide a proof based on symmetric tensor-network formulation)

# A Corollary

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\ {}^aU_b = \alpha(a,b)U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

**Corollary**: Assuming $G = G_1 \times Z_N$ , where $Z_N = \{\mathbf{I}, g, g^2 ... g^{N-1}\}$ ,

A sym-SRE phase is realizable if and only if the following two conditions are both satisified, and the realizable sym-SRE phase must be an SPT.

(1) $\alpha^N \simeq \mathbf{1} \in H^2(G, U(1))$

(2) $\gamma_g^\alpha \simeq \mathbf{1} \in H^1(G, U\tilde{(}1))$ where $\gamma_g^\alpha(a) \equiv \frac{\alpha(g,a)}{\alpha(a,g)}, \forall a \in G$

# A Corollary

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a \, {}^a U_b = \alpha(a,b) U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

**Corollary**: Assuming $G = G_1 \times Z_N$ , where $Z_N = \{\mathbf{I}, g, g^2 ... g^{N-1}\}$ ,

A sym-SRE phase is realizable if and only if the following two conditions are both satisified, and the realizable sym-SRE phase must be an SPT.

(1) $\alpha^N \simeq \mathbf{1} \in H^2(G, U(1))$

(2) $\gamma_g^\alpha \simeq \mathbf{1} \in H^1(G, \tilde{U(1)})$ where $\gamma_g^\alpha(a) \equiv \frac{\alpha(g,a)}{\alpha(a,g)}, \forall a \in G$

The first condition is well anticipated: in the enlarged unit cell one needs to have regular representation, otherwise HOLSM forbids sym-SRE phases.

# A Corollary

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\ {}^aU_b = \alpha(a,b)U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

**Corollary**: Assuming $G = G_1 \times Z_N$ , where $Z_N = \{\mathbf{I}, g, g^2 ... g^{N-1}\}$ ,

A sym-SRE phase is realizable if and only if the following two conditions are both satisified, and the realizable sym-SRE phase must be an SPT.

(1) $\alpha^N \simeq \mathbf{1} \in H^2(G, U(1))$

(2) $\gamma_g^\alpha \simeq \mathbf{1} \in H^1(G, U(\tilde{1}))$ where $\gamma_g^\alpha(a) \equiv \frac{\alpha(g,a)}{\alpha(a,g)}, \forall a \in G$

The second condition is not obvious from physical point of view.

e.g.: N=2 and $G_1 = \tilde{Z}_2$, $\alpha$ is the projective rep. of $G = \tilde{Z}_2 \times Z_2$, like a spin-1/2.

Then (1) is satisfied but (2) is violated. sym-SRE phase is impossible.

# New HOLSM-type constraint

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\ {}^aU_b = \alpha(a,b)U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

**Corollary**: Assuming $G = G_1 \times Z_N$ , where $Z_N = \{\mathbf{I}, g, g^2 ... g^{N-1}\}$ ,

A sym-SRE phase is realizable if and only if the following two conditions are both satisified, and the realizable sym-SRE phase must be an SPT.

(1) $\alpha^N \simeq \mathbf{1} \in H^2(G, U(1))$

(2) $\boxed{\gamma_g^\alpha \simeq \mathbf{1} \in H^1(G, \tilde{U}(1)) \text{ where } \gamma_g^\alpha(a) \equiv \frac{\alpha(g,a)}{\alpha(a,g)}, \forall a \in G}$

The second condition is not obvious from physical point of view.

e.g.: N=2 and $G_1 = \tilde{Z}_2$, $\alpha$ is the projective rep. of $G = \tilde{Z}_2 \times Z_2$, like a spin-1/2.

Then (1) is satisfied but (2) is violated. sym-SRE phase is impossible.

even when the HOLSM is silent, there is a new constraint forbidding sym-SRE

# A Corollary

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\ {}^aU_b = \alpha(a,b)U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

**Corollary**: Assuming $G = G_1 \times Z_N$ , where $Z_N = \{\mathbf{I}, g, g^2 ... g^{N-1}\}$ ,

A sym-SRE phase is realizable if and only if the following two conditions are both satisified, and the realizable sym-SRE phase must be an SPT.

(1) $\alpha^N \simeq \mathbf{1} \in H^2(G, U(1))$

(2) $\gamma_g^\alpha \simeq \mathbf{1} \in H^1(G, \tilde{U(1)})$ where $\gamma_g^\alpha(a) \equiv \frac{\alpha(g,a)}{\alpha(a,g)}, \forall a \in G$

Applications: N=2 and $G_1 = Z_2^T$: time-reversal

(a) One Kramer doublet per unit cell → SPT in which Ising defect is Kramer

(b) One non-Kramer doublet per unit cell → Levin-Gu $Z_2$ SPT.

We also provide exactly solvable decorated quantum dimer models realizing these symmetry-enforced SPTs.

# The question

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\ {}^aU_b = \alpha(a,b)U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

We ask the following question:

Is a gapped SRE liquid (sym-SRE) phase possible? If the answer is yes, what kinds of sym-SRE phases are realizable?

PLAN:

1. Main results: a theorem
2. A corollary: new HOLSM-type constraint
3. Sketch of the proof
4. Model realizations (A simple model realizing SPT phase)

# Sketch of the proof

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a \, {}^a U_b = \alpha(a,b) U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

Theorem in physics language: a sym-SRE phase is realizable if and only if its g-symmetry-defect carries the projective representation $\alpha$ .

# Sketch of the proof

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\, {}^aU_b = \alpha(a,b)U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

Theorem in physics language: a sym-SRE phase is realizable if and only if its g-symmetry-defect carries the projective representation $\alpha$ .

- "only if" part: an entanglement pumping argument

  (similar to arguments by Lu/Ran/Oshikawa and Wu/Ho/Lu 2017.)

- "if" part is more challenging: we need to provide generic constructions of these symmetry-enforced SPTs. We achieve this using a recently developed symmetric tensor-network formulation. (Jiang and Ran, 2017)

# Sketch of the proof

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a \, {}^a U_b = \alpha(a,b) U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

Theorem in physics language: a sym-SRE phase is realizable if and only if its g-symmetry-defect carries the projective representation $\alpha$ .

- "only if" part: an entanglement pumping argument

(similar to arguments by Lu/Ran/Oshikawa and Wu/Ho/Lu 2017.)

- "if" part is more challenging: we need to provide generic constructions of these symmetry-enforced SPTs. We achieve this using a recently developed symmetric tensor-network formulation. (Jiang and Ran, 2017)

# Sketch of the proof

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\,{}^aU_b = \alpha(a,b)U_{ab}$

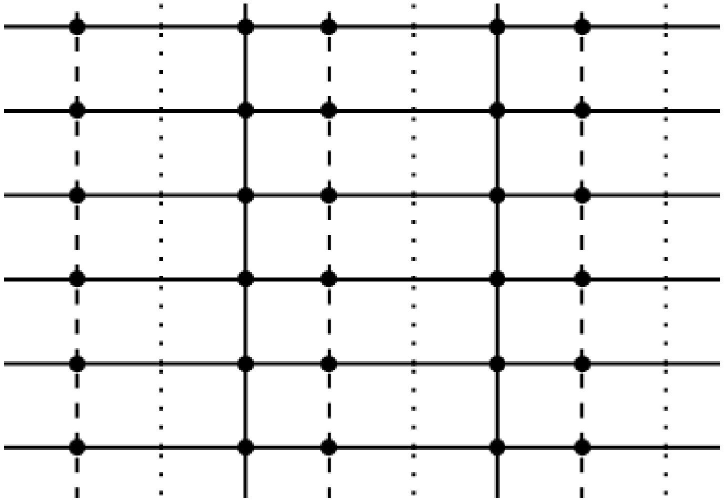
and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

Theorem in physics language: a sym-SRE phase is realizable if and only if its g-symmetry-defect carries the projective representation $\alpha$ .

"only if" part: entanglement pumping:

Long cylinder sample with $L_y = k \cdot N + 1$.



# Sketch of the proof

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\ {}^aU_b = \alpha(a,b)U_{ab}$

and respecting magnetic translation symmetry:

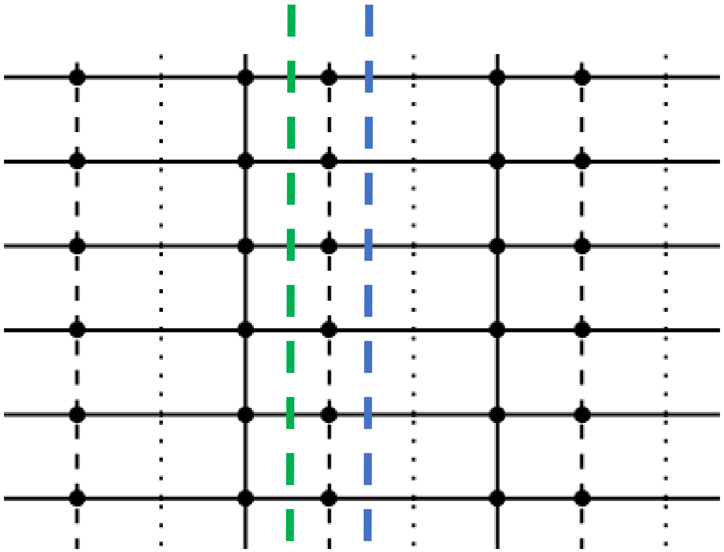
$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

Theorem in physics language: a sym-SRE phase is realizable if and only if its g-symmetry-defect carries the projective representation $\alpha$ .

"only if" part: entanglement pumping:
Long cylinder sample with $L_y = k \cdot N + 1$.

- In a sym-SRE state, the proj. rep. of entanglement eigenstates at a given cut is fixed.

Green cut and blue cut differ by a projective rep. $\alpha$



# Sketch of the proof

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a \, {}^{a}U_b = \alpha(a,b) U_{ab}$

and respecting magnetic translation symmetry:

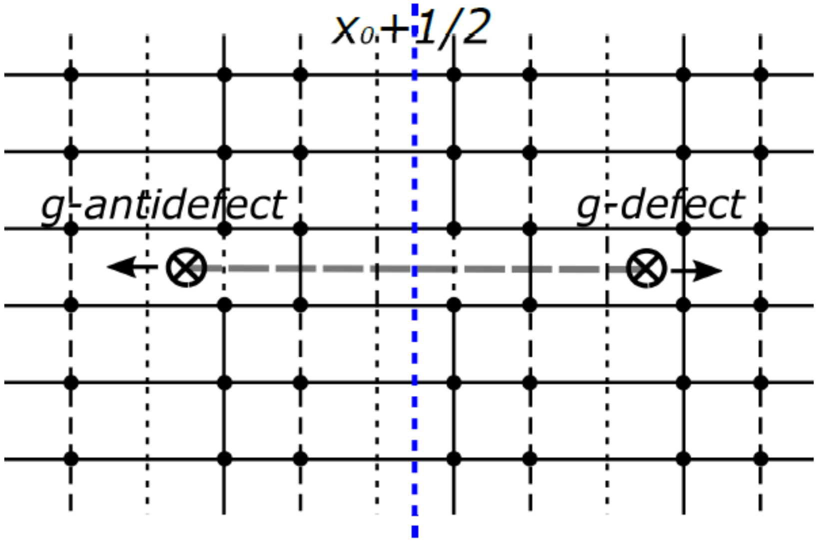
$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

Theorem in physics language: a sym-SRE phase is realizable if and only if its g-symmetry-defect carries the projective representation $\alpha$ .

"only if" part: entanglement pumping:

Long cylinder sample with $L_y = k \cdot N + 1$.

- In a sym-SRE state, the proj. rep. of entanglement eigenstates at a given cut is fixed.
- After separating one pair of defects per row, the net effect is $T_x^{orig.}$.
  → The g-defect must carry $\alpha$.



# Sketch of the proof

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\ {}^aU_b = \alpha(a,b)U_{ab}$

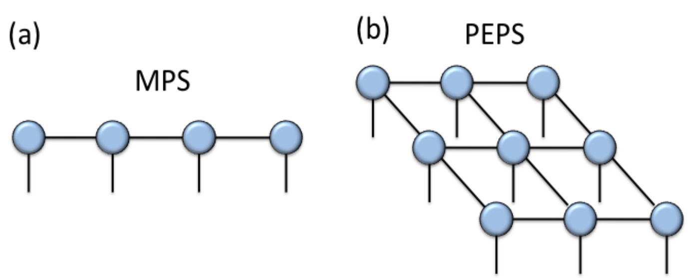
and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

Theorem in physics language: a sym-SRE phase is realizable if and only if its g-symmetry-defect carries the projective representation $\alpha$ .

"if" part: symmetric tensor-network construction

**1D-MPS, 2D-PEPS, and 3D generalizations**

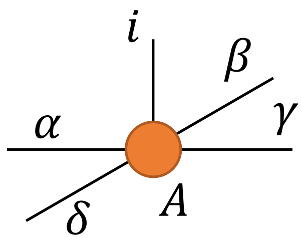



figures from R. Orus,
Annals Phys. (2014)

Cirac, Verstraete, Vidal, Gu, Levin, Wen, White, Xiang….

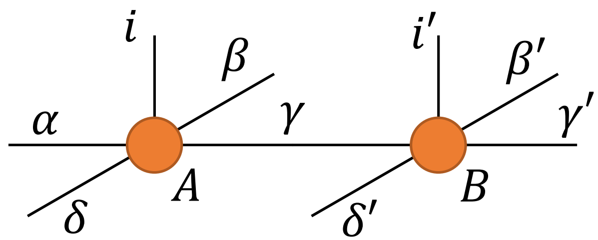
# A brief introduction to tensor-networks

tensor



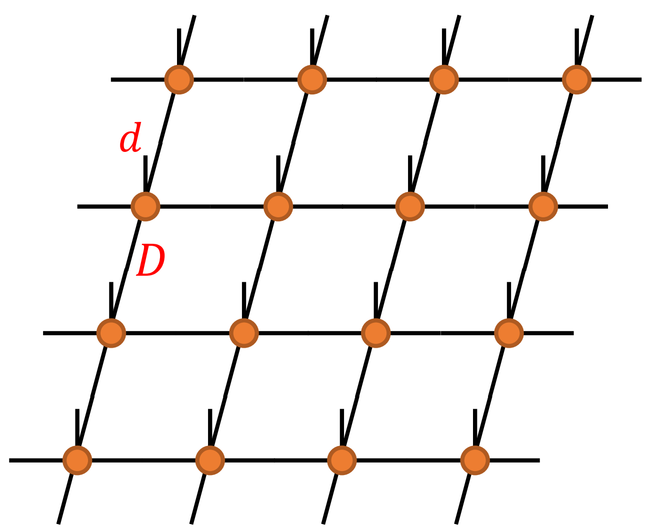

$$= A^{i}_{\alpha\beta\gamma\delta} \sim \sum A^{i}_{\alpha\beta\gamma\delta}|i\rangle \otimes |\alpha\beta\gamma\delta\rangle$$

tensor contraction

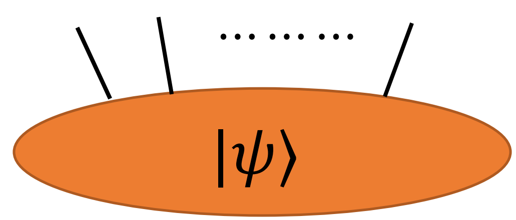



$$= \sum_{\gamma} A^{i}_{\alpha\beta\gamma\delta} \cdot B^{i'}_{\gamma\beta'\gamma'\delta'}$$

PEPS



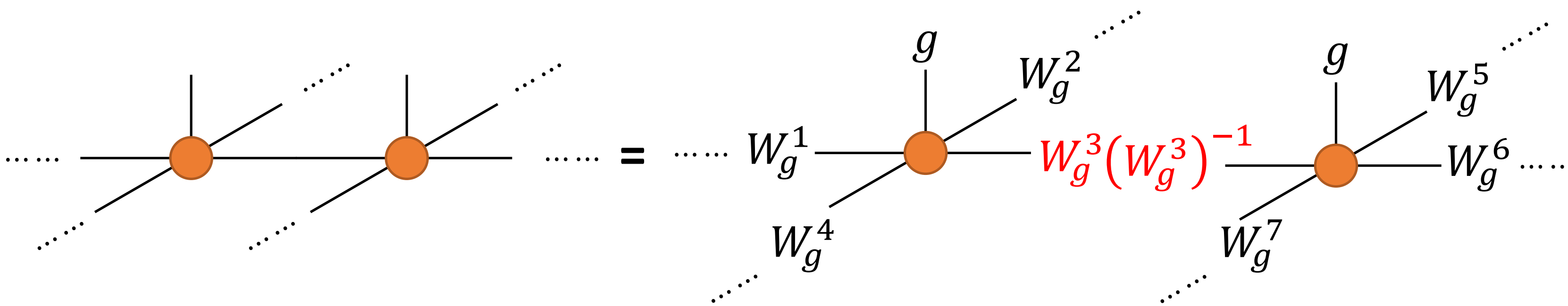

$$|\psi\rangle = \sum_{\{i\}} c_{i_1 i_2 \dots i_n} |i_1, i_2, \dots, i_n\rangle$$



# Symmetric tensor-networks

$$|\psi\rangle = g|\psi\rangle$$

global symmetries on physical wavefunctions ~ gauge transformation on internal legs

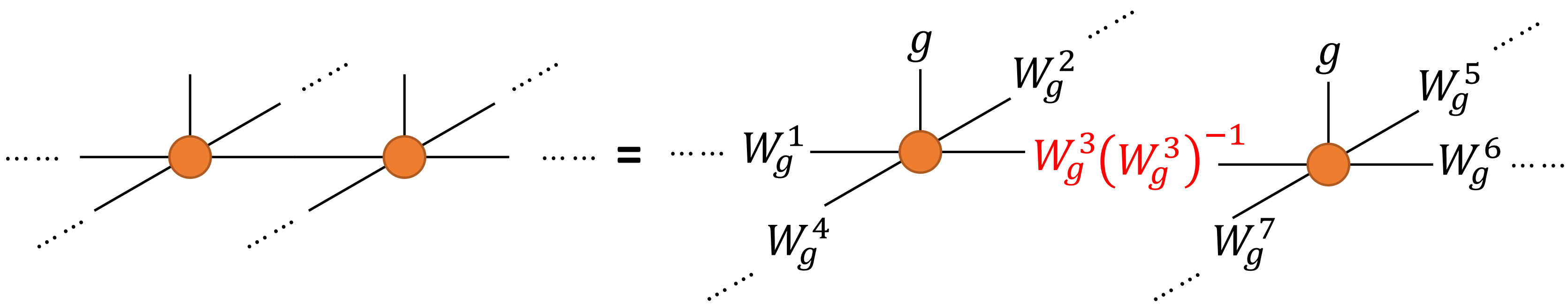



$$TN = W_g g \circ TN$$

# Symmetric tensor-networks

$$|\psi\rangle = g|\psi\rangle$$

global symmetries on physical wavefunctions ~ gauge transformation on internal legs




$$TN = W_g g \circ TN$$

Jiang&YR, 2015, 2016

The consistency algebraic conditions for $W_g$ can characterize SPT phases.
(mathematically: crossed-module extension)
Main advantage: onsite symmetry and spatial symmetry are treated on same footing.

# Sketch of the proof

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\ {}^aU_b = \alpha(a,b)U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

Theorem in physics language: a sym-SRE phase is realizable if and only if its g-symmetry-defect carries the projective representation $\alpha$ .

"if" part: symmetric tensor-network construction

# Sketch of the proof

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\,{}^{a}U_b = \alpha(a,b)U_{ab}$

and respecting magnetic translation symmetry:

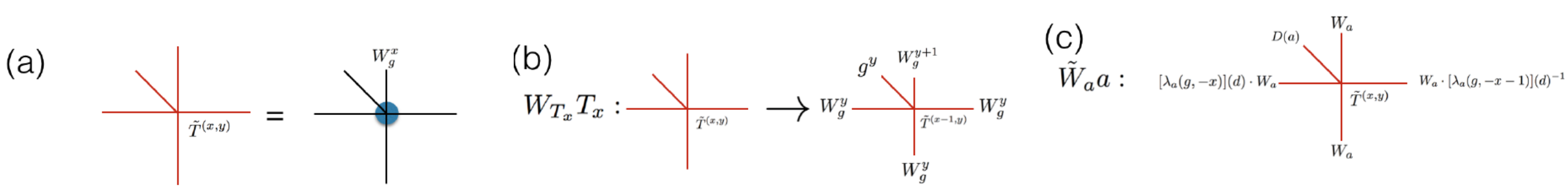
$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

Theorem in physics language: a sym-SRE phase is realizable if and only if its g-symmetry-defect carries the projective representation $\alpha$ .

"if" part: symmetric tensor-network construction

Given a symmetric tensor-network SPT state with a regular representation per unit cell and Respecting the usual translation symmetry, and its g-defect carries projective representation $\alpha$ ,

there is a prescription to modify it into the tensor-network SPT state with a projective representation $\alpha$ per unit cell and respecting the magnetic translation symmetry.



# The question

Consider 2+1D bosonic systems with an onsite symmetry group G, having one projective representation $\alpha$ of G per unit cell: $U_a\,{}^aU_b = \alpha(a,b)U_{ab}$

and respecting magnetic translation symmetry:

$$T_x T_y T_x^{-1} T_y^{-1} = g \qquad g \in \text{ Center of } G$$

We ask the following question:

Is a gapped SRE liquid (sym-SRE) phase possible? If the answer is yes, what kinds of sym-SRE phases are realizable?

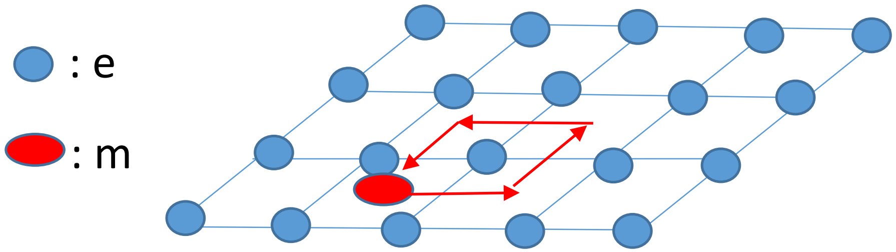
PLAN:

1. Main results: a theorem
2. A corollary: new HOLSM-type constraint
3. Sketch of the proof
4. Model realizations (A simple model realizing SPT phase)

# Another picture of symmetry-enforced SPT

Let us consider a familiar situation: G=SO(3)

A Z2 (toric-code topological order) spin liquid with a spin-1/2 per unit cell respecting regular translation symm.




Quite generically: (Cheng, Zaletel, Barkeshli, Vishwanath, Bonderson,2015)

the spinon e carries the spin-1/2,

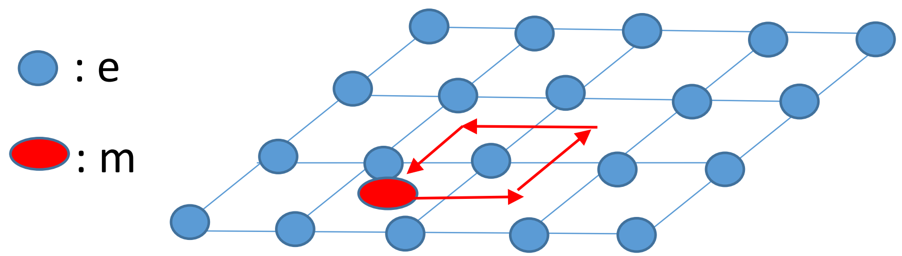
the vison m has nontrivial translation symmetry fractionalization:

$$T_x T_y T_x^{-1} T_y^{-1}[m] = -1$$

# Another picture of symmetry-enforced SPT

Let us consider a familiar situation: G=SO(3)

A Z2 (toric-code topological order) spin liquid with a spin-1/2 per unit cell respecting regular translation symm.




Quite generically: (Cheng, Zaletel, Barkeshli, Vishwanath, Bonderson,2015)

the spinon e carries the spin-1/2,

the vison m has nontrivial translation symmetry fractionalization:

$$T_x T_y T_x^{-1} T_y^{-1}[m] = -1$$

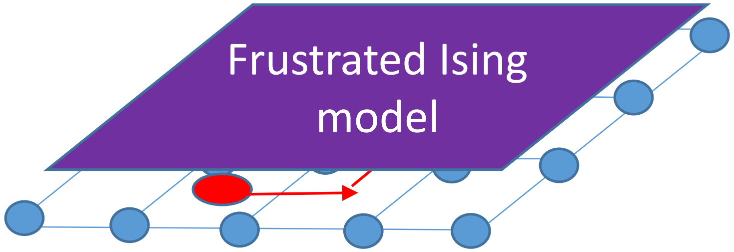
Consistent with HOLSM, in order to kill the topological order, there is no way to condense either e or m without breaking symmetry

# Another picture of symmetry-enforced SPT

What if G=SO(3)xIsing ?

A Z2 (toric-code topological order) spin liquid with a spin-1/2 per unit cell respecting Ising magnetic translation symm:

$$T_x T_y T_x^{-1} T_y^{-1} = \text{Ising}$$



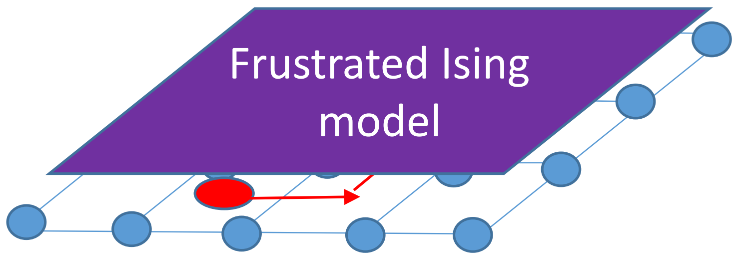

e.g., the previous Z2 spin liquid stacked with a layer of frustrated Ising model

# Another picture of symmetry-enforced SPT

What if G=SO(3)xIsing ?

A Z2 (toric-code topological order) spin liquid with a spin-1/2 per unit cell respecting Ising magnetic translation symm:

$$T_x T_y T_x^{-1} T_y^{-1} = \mathrm{Ising}$$




e.g., the previous Z2 spin liquid stacked with a layer of frustrated Ising model

the spinon e carries the spin-1/2,

But the vison m can have trivial symmetry fractionalization:

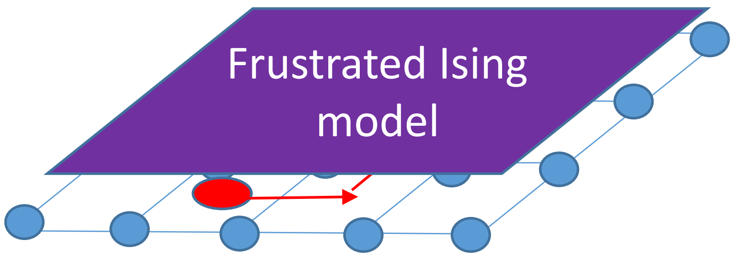
$$T_x T_y T_x^{-1} T_y^{-1}[m] = -1 = \mathrm{Ising}[m]$$

# Another picture of symmetry-enforced SPT

What if G=SO(3)xIsing ?

A Z2 (toric-code topological order) spin liquid with a spin-1/2 per unit cell respecting Ising magnetic translation symm:

$$T_x T_y T_x^{-1} T_y^{-1} = \text{Ising}$$




e.g., the previous Z2 spin liquid stacked with a layer of frustrated Ising model

the spinon e carries the spin-1/2,

But the vison m can have trivial symmetry fractionalization:

$$T_x T_y T_x^{-1} T_y^{-1}[m] = -1 = \text{Ising}[m]$$

Condensing such Ising-odd vison does not break physical symmetry and kills topological order
The resulting sym-SRE phase MUST be an SPT. (anyon-condensation mechanism, Jiang&YR 2016 )

# The anyon condensation mechanism to obtain SPT

an SET phase $\xrightarrow{\text{Condense certain fluxes}}$ an SPT phase

- Gauge group: $Z_{N_1} \times Z_{N_2} \times \cdots$ & symmetry group: $SG$

Jiang&YR, 2016

- e-particles feature nontrivial symmetry fractionalization

- m-particles have trivial fractionalization, but can carry usual quantum numbers

$\Omega_{g_1} \cdot \Omega_{g_2} = \boldsymbol{\lambda}(\boldsymbol{g_1}, \boldsymbol{g_2}) \cdot \Omega_{g_1 g_2}$ $\Omega_g$ ~ symmetry defect, $\lambda$ ~ certain $m$ particle

- Condensing m-particles without breaking symmetry, which requires:

  1. Condensed $m$'s carry 1D symmetry irrep: $\boldsymbol{\chi_m}(\boldsymbol{g})$

  2. $\chi_m(g) \cdot \chi_{m'}(g) = \chi_{mm'}(g)$

These results are also obtained by symmetric tensor-network formulation

- After condensing those $m$'s, we get an sym-SRE phase with

$$\omega(g_1, g_2, g_3) \equiv \chi_{\lambda(g_2,g_3)}(g_1), \qquad [\omega] \in H^3(SG, U(1))$$

# A somewhat simple model realizing symm-enforced SPT

- Following the anyon-condensation mechanism, we can design somewhat simple models realizing bosonic SPT phases. (need 3-spin interactions)

- The model looks like this:

Global symmetry: U(1) x Ising

$$H = H_{U(1)} + H_{Ising} + \lambda \cdot W$$

Condensing
Ising-odd m-particle

0 SET: $Z_2$ gauge
e carry ½ U(1) charge

$\lambda c$

SPT: Ising-defect
carry ½ U(1) charge

$\lambda$

U(1)-Layer

$W$

Ising-Layer

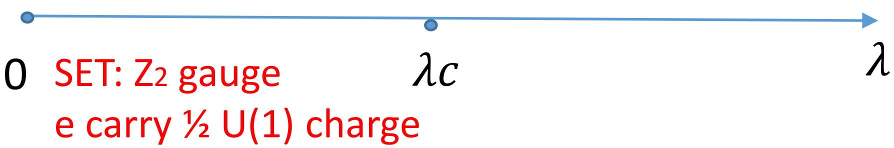
$$H_{Ising} = h \cdot \Sigma \sigma^x$$

# A somewhat simple model realizing symm-enforced SPT

Global symmetry: U(1) x Ising

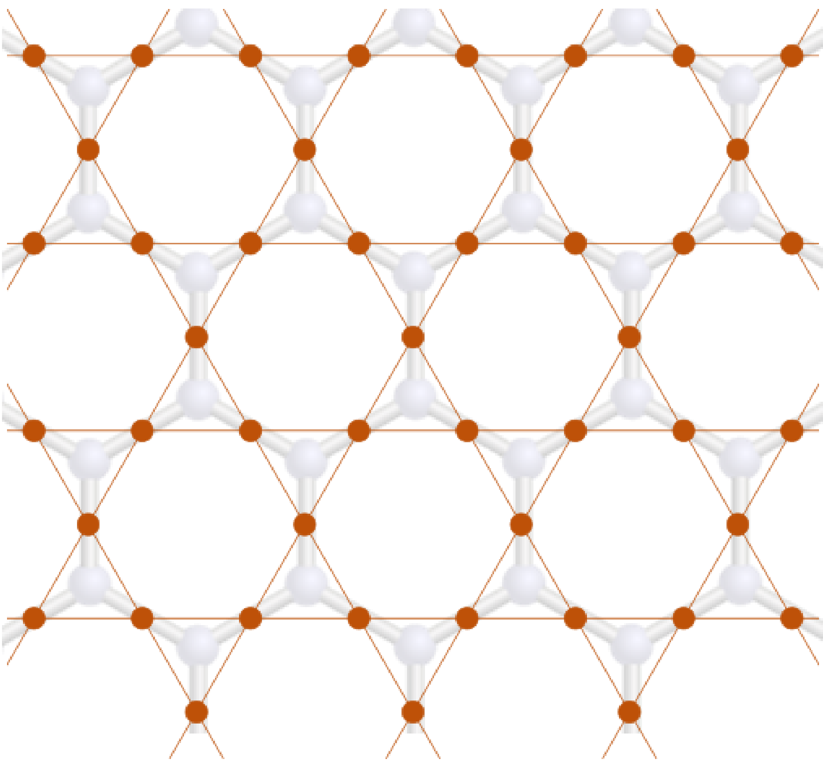
$$H = H_{U(1)} + H_{Ising} + \lambda \cdot W$$

U(1)-layer: Half-filled hard-core bosons on the kagome lattice




$$H_{U(1)} = -t\Sigma b_i^{+} b_j + V_1 \Sigma n_i n_j + V_2 \Sigma n_i n_j + V_3 \Sigma n_i n_j$$

$$t \ll V_1 = V_2 = V_3 = V$$

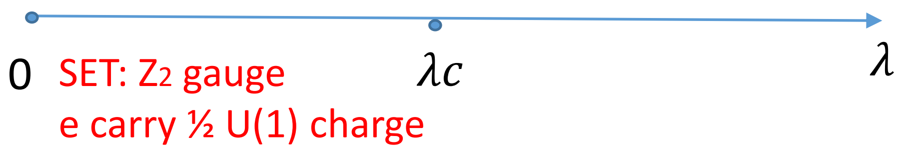


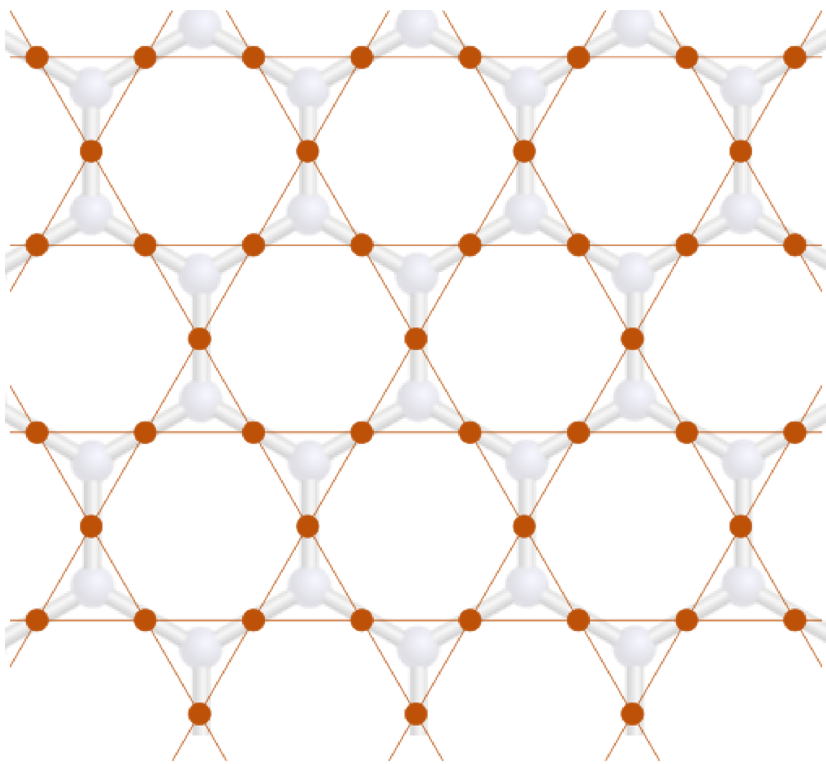
In this regime, $H_{U(1)}$ is in a deconfined $Z_2$ spin liquid phase:
e-particle carries ½ U(1)-charge.
(Balents,Fisher,Girvin 2001)

# A somewhat simple model realizing symm-enforced SPT

Global symmetry: U(1) x Ising

$$H = H_{U(1)} + H_{Ising} + \lambda \cdot W$$



U(1)-layer: Half-filled hard-core bosons on the kagome lattice

$$H_{U(1)} = -t\Sigma b_i^+ b_j + V_1 \Sigma n_i n_j + V_2 \Sigma n_i n_j + V_3 \Sigma n_i n_j$$

$$t \ll V_1 = V_2 = V_3 = V$$

Ising-layer: transverse field Ising spins on the honeycomb lattice

$$H_{Ising} = h \cdot \Sigma \sigma^x$$

$$h \ll V$$

# A somewhat simple model realizing symm-enforced SPT

Global symmetry: U(1) x Ising

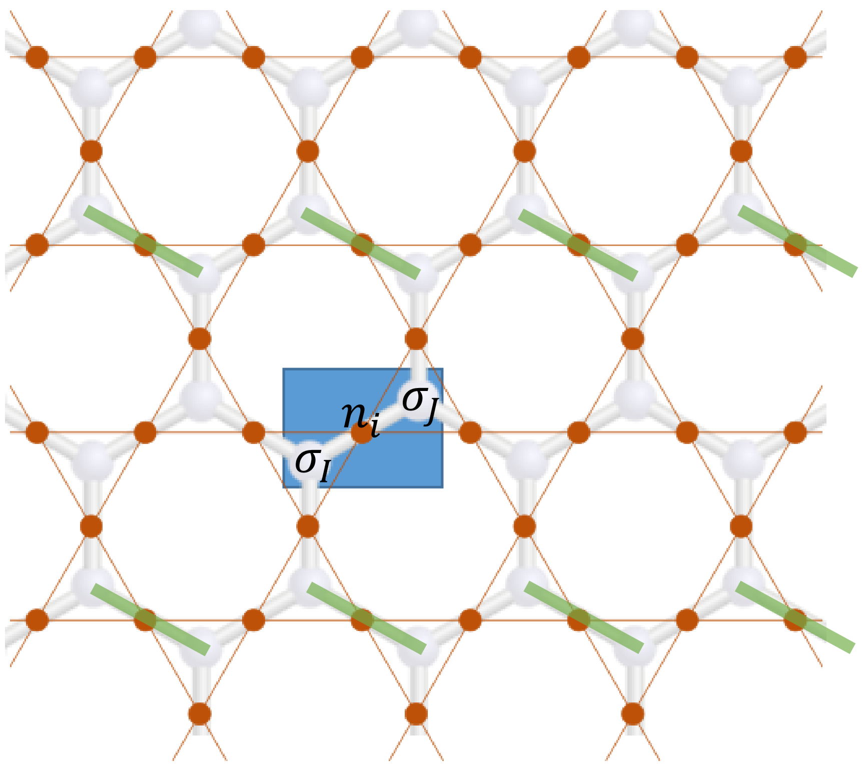
Condensing
Ising-odd m-particle

0 SET: $Z_2$ gauge
e carry ½ U(1) charge

$\lambda c$

SPT phase

$\lambda$




$$H = H_{U(1)} + H_{Ising} + \lambda \cdot W$$

U(1)-layer: Half-filled hard-core bosons
on the kagome lattice

$$H_{U(1)} = -t\Sigma b_i^+ b_j + V_1 \Sigma n_i n_j$$
$$+V_2 \Sigma n_i n_j + V_3 \Sigma n_i n_j$$

$$t \ll V_1 = V_2 = V_3 = V$$

Ising-layer: transverse field Ising spins
on the honeycomb lattice

$$H_{Ising} = h \cdot \Sigma \sigma^x$$

$$h \ll V$$

$\lambda \cdot W$: 3-spin interaction coupling two layers

$$\lambda \cdot W = \lambda \cdot \sum (n_i - 1/2) \cdot (s_{IJ} \sigma_I^z \sigma_J^z)$$

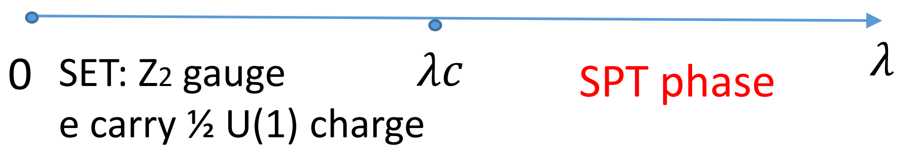
$s_{IJ}$=-1 on green bonds, $s_{IJ}$=+1 otherwise

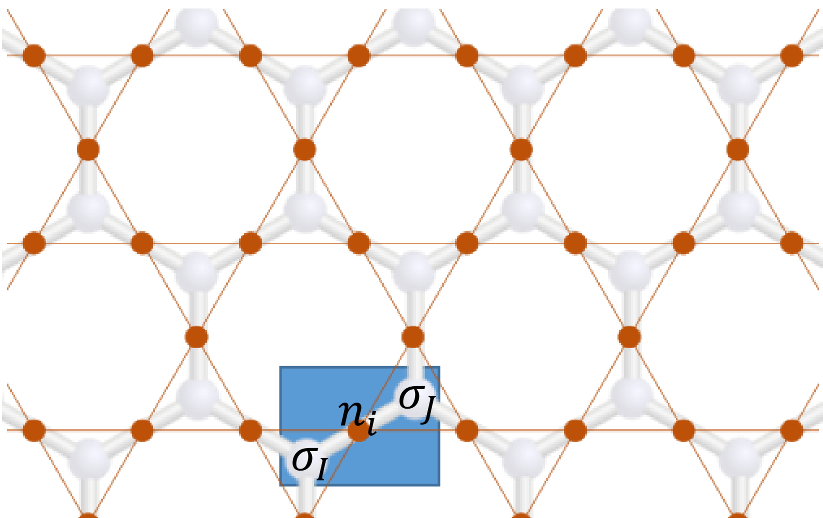
Ising magnetic translation symmetric

# A somewhat simple model realizing symm-enforced SPT

Global symmetry: U(1) x Ising

Condensing
Ising-odd m-particle




One can analytically show:
SPT phase is realized when
$t, h \ll \lambda \ll$V and $\frac{t}{V} \ll \frac{h^2}{\lambda^2}$

$$H = H_{U(1)} + H_{Ising} + \lambda \cdot W$$

U(1)-layer: Half-filled hard-core bosons on the kagome lattice

$$H_{U(1)} = -t\Sigma b_i^+ b_j + V_1 \Sigma n_i n_j + V_2 \Sigma n_i n_j + V_3 \Sigma n_i n_j$$

$$t \ll V_1 = V_2 = V_3 = V$$

Ising-layer: transverse field Ising spins on the honeycomb lattice

$$H_{Ising} = h \cdot \Sigma \sigma^x$$

$$h \ll V$$

$\lambda \cdot W$: 3-spin interaction coupling two layers

$$\lambda \cdot W = \lambda \cdot \sum (n_i - 1/2) \cdot (s_{IJ} \sigma_I^z \sigma_J^z)$$

$s_{IJ}$=-1 on green bonds, $s_{IJ}$=+1 otherwise
Ising magnetic translation symmetric

# Summary

- For 2+1D bosonic systems with proj. rep. per unit cell respecting magnetic translation symmetry, we give sufficient and necessary condition for a sym-SRE phase to exist.
- If such sym-SRE phase exist, it must be SPT (symmetry-enforced SPT). All realizable SPT phases form a coset structure.
- Sometimes such sym-SRE does not exist due to nonobvious reason:

  new HOLSM-type constraint
- Simple Model realizations of SPT (via anyon condensation mechansim)

# Summary

- For 2+1D bosonic systems with proj. rep. per unit cell respecting magnetic translation symmetry, we give sufficient and necessary condition for a sym-SRE phase to exist.
- If such sym-SRE phase exist, it must be SPT (symmetry-enforced SPT). All realizable SPT phases form a coset structure.
- Sometimes such sym-SRE does not exist due to nonobvious reason:

  new HOLSM-type constraint
- Simple Model realizations of SPT (via anyon condensation mechansim)

Thank you!