# *Duality between 2+1d Quantum Critical Points, Theory and Numerics*

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## *Duality between 2+1d Quantum Critical Points, Theory and Numerics*

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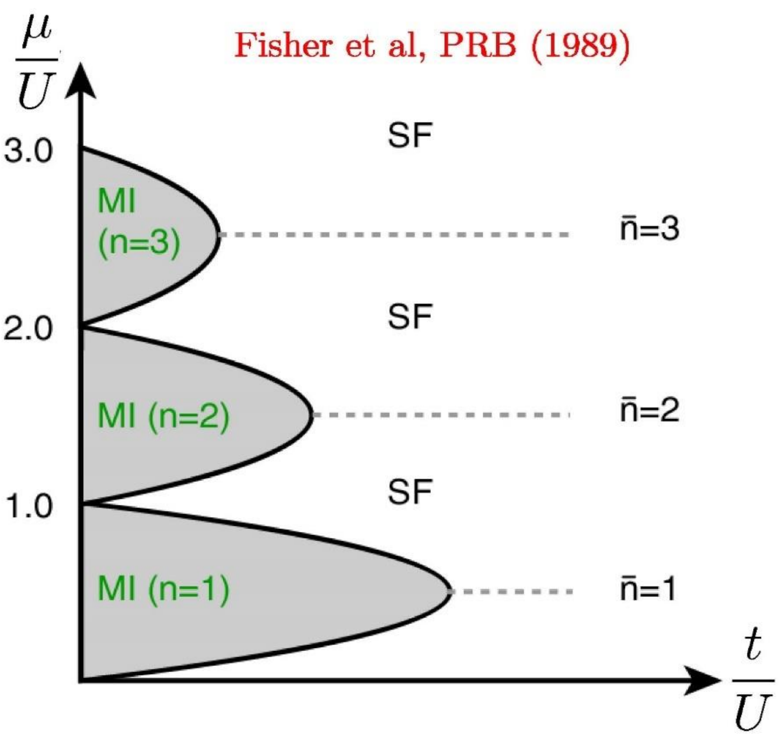
Main references:
arXiv:1510.06032, Xu, You
arXiv:1703.02426, Wang, Nahum, Metlitski, Xu, Senthil
arXiv:1705.10670, Qin, He, You, Lu, Sen, Sandvik, Xu, Meng

**2+1d CFTs, usually occur at quantum critical points**

Classic example: quantum critical point between superfluid and Bose Mott insulator.



## *Introduction*

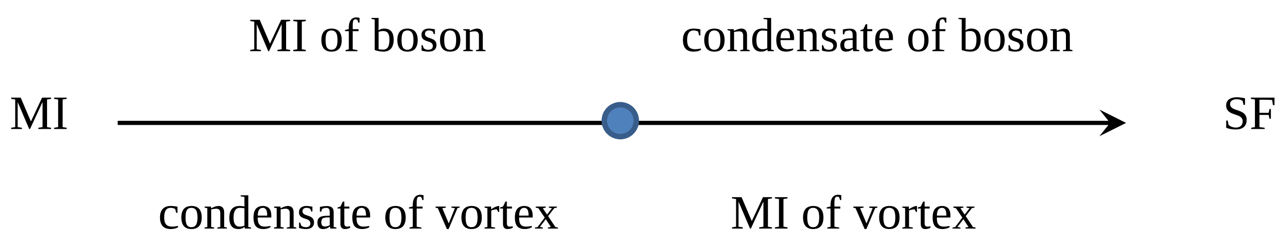
### Challenge of studying 2+1d CFTs:

Standard methods of studying 2+1d CFTs are, $1/N$ expansion, epsilon expansion, both have difficulty of convergence for 2+1d CFTs.

### A powerful nonperturbative method: duality

Duality maps an unknown problem to a (hopefully) known problem.

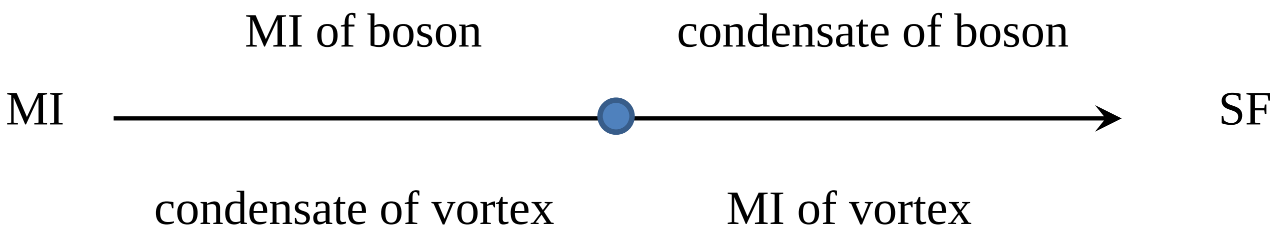
Classic example: Particle-Vortex duality (Peskin, 1978, Dastupta, Halperin, 1981, Fisher, Lee, 1989, can be “derived” on the lattice):



## *Introduction*

Classic example: Boson-Vortex duality (Peskin, 1978, Dastupta, Halperin, 1981, Fisher, Lee, 1989 , can be “derived” on the lattice):

$$\mathcal{L} = |\partial_\mu \Phi|^2 + r|\Phi|^2 + g|\Phi|^4$$




$$\mathcal{L} = |(\partial_\mu - i a_\mu)\tilde{\Phi}|^2 - r|\tilde{\Phi}|^2 + g|\tilde{\Phi}|^4$$

# *Introduction*

Example 2: 3d Dirac fermion and bosonic QED with Chern-Simons term at level-1 (Chen, Fisher, Wu 1993, recent lattice version of the duality: Chen, et.al. arXiv:1705.05841):

$$\mathcal{L} = |(\partial_\mu - i a_\mu)\Phi|^2 + r|\Phi|^2 + g|\Phi|^4 + \frac{i}{4\pi} a \wedge da - \frac{i}{2\pi} a \wedge dA$$

$r < 0$ Trivial insu. Integer QH $r > 0$

$m < 0$ $m > 0$

$$\mathcal{L} = \bar{\psi}\gamma_\mu(\partial_\mu - iA_\mu)\psi + m\bar{\psi}\psi + \bar{\Psi}\gamma_\mu(\partial_\mu - iA_\mu)\Psi + M\bar{\Psi}\Psi$$

$$\mathcal{L} = \bar{\psi}\gamma_\mu(\partial_\mu - iA_\mu)\psi + m\bar{\psi}\psi - \frac{i}{8\pi} A \wedge dA$$

## *Introduction*

Example 4: fermionic particle-vortex duality: (Son, 2015, Metlitski, Vishwanath, 2015, Wang, Senthil, 2015)

$$\mathcal{L} = \bar{\chi}\gamma_\mu(\partial_\mu - iA_\mu)\chi + m\bar{\chi}\chi$$

$m < 0$ +1/2 QH -1/2 QH $m > 0$

$\tilde{m} > 0$ $\tilde{m} < 0$

$$\mathcal{L} = \bar{\psi}\gamma_\mu(\partial_\mu - ia_\mu)\psi + \frac{i}{4\pi} a \wedge dA + \tilde{m}\bar{\psi}\psi$$

(1), the +-1/2 QH state is the anomalous effect at the boundary of the 3d TI; (2), *a* only allows 4*πn* flux;

(3), These basic dualities can lead to a large web of dualities, some of these dualities are of great importance to CMT.

## *Self-dual N=2 QED3*

Restating the conjecture: the *N*=1 QED3 flows to an IR fixed point, which is equivalent to a noninteracting Dirac fermion.

$$\mathcal{L} = \bar{\chi}\gamma_\mu(\partial_\mu - iA_\mu)\chi$$

$$\leftrightarrow \ \mathcal{L} = \bar{\psi}\gamma_\mu(\partial_\mu - ia_\mu)\psi + \frac{i}{4\pi} a \wedge dA$$

Assuming this is true, we can derive the following descendant duality: The *N*=2 QED3, if it is a CFT, is self-dual, Xu, You, arXiv:1510.06032

$$\mathcal{L} = \sum_{j=1}^{2} \bar{\chi}_j\gamma_\mu(\partial_\mu - ia_\mu)\chi_j + iA_\mu\bar{\chi}\gamma_\mu\tau^z\chi + \frac{i}{2\pi} a \wedge dB$$

$$\leftrightarrow \ \mathcal{L} = \sum_{j=1}^{2} \bar{\psi}_j\gamma_\mu(\partial_\mu - ib_\mu)\psi_j + iB_\mu\bar{\psi}\gamma_\mu\tau^z\psi + \frac{i}{2\pi} b \wedge dA$$

# *Self-dual N=2 QED3*

Deriving the self-duality (Xu, You, arXiv:1510.06032):

$$\mathcal{L} = \sum_{j=1}^{2} \bar{\chi}_j \gamma_\mu (\partial_\mu - i a_\mu) \chi_j + i A_\mu \bar{\chi} \gamma_\mu \tau^z \chi + \frac{i}{2\pi} a \wedge dB$$

Step 1: run the fermion-fermion duality for each flavor:

$$\begin{aligned} \leftrightarrow \ \mathcal{L} = & \bar{\psi}_1 \gamma_\mu (\partial_\mu - i b_\mu) \psi_1 + \bar{\psi}_2 \gamma_\mu (\partial_\mu - i c_\mu) \psi_2 \\ & - \frac{i}{4\pi} a \wedge d(b + c - 2B) - \frac{i}{4\pi} A \wedge d(b - c) \end{aligned}$$

Step 2: Integrating out dynamical gauge field *a*:

$$\leftrightarrow \ \mathcal{L} = \sum_{j=1}^{2} \bar{\psi}_j \gamma_\mu (\partial_\mu - i b_\mu) \psi_j + i B_\mu \bar{\psi} \gamma_\mu \tau^z \psi + \frac{i}{2\pi} b \wedge dA$$

Other Derivation of this duality: Mross, et.al. arXiv:1510.08455
Karch, Tong, arXiv:1606.01893, Hsin, Seiberg, arXiv:1607.07457

## *Self-dual N=2 QED3*

$$\mathcal{L} = \sum_{j=1}^{2} \bar{\chi}_j \gamma_\mu (\partial_\mu - i a_\mu) \chi_j + i A_\mu \bar{\chi} \gamma_\mu \tau^z \chi + \frac{i}{2\pi} a \wedge dB$$

$$\leftrightarrow \ \mathcal{L} = \sum_{j=1}^{2} \bar{\psi}_j \gamma_\mu (\partial_\mu - i b_\mu) \psi_j + i B_\mu \bar{\psi} \gamma_\mu \tau^z \psi + \frac{i}{2\pi} b \wedge dA$$

Comments:

1, the *N*=2 QED3 has O(4) symmetry. The SO(4) ~ SU(2) x SU(2) is the flavor symmetry of both $\psi$ and $\chi$. The $Z_2$ subgroup of O(4) is the self-duality transformation.
There is another way to see the O(4) symmetry, by mapping this model to a low-energy effective field theory in terms of gauge invariant O(4) vector boson (Senthil, Fisher 2005).

## *Self-dual N=2 QED3*

$$\mathcal{L} = \sum_{j=1}^{2} \bar{\chi}_j \gamma_\mu (\partial_\mu - i a_\mu) \chi_j + i A_\mu \bar{\chi} \gamma_\mu \tau^z \chi + \frac{i}{2\pi} a \wedge dB$$

$$\leftrightarrow \ \mathcal{L} = \sum_{j=1}^{2} \bar{\psi}_j \gamma_\mu (\partial_\mu - i b_\mu) \psi_j + i B_\mu \bar{\psi} \gamma_\mu \tau^z \psi + \frac{i}{2\pi} b \wedge dA$$

Comments:

2, there is an O(4) breaking but SO(4) invariant relevant perturbation:

$$m\bar{\psi}\psi \sim -m\bar{\chi}\chi$$

Tuning *m* drives a transition from a bosonic SPT state to a trivial state (Grover, Vishwanath, 2012, Lu, Lee, 2012):
the Chern-Simons level of *A* and *B* changes by +2 and -2 respectively.
In this case, there are extra background terms $\mathrm{CS}_1[A] + \mathrm{CS}_{-1}[B]$

## *Self-dual N=2 QED3*

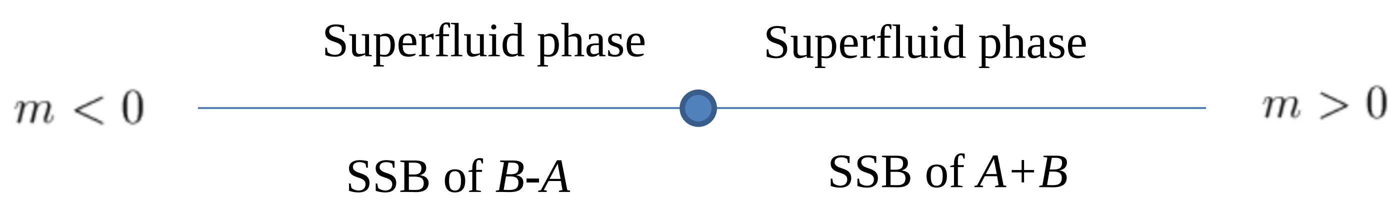
$$\mathcal{L} = \sum_{j=1}^{2} \bar{\chi}_j \gamma_\mu (\partial_\mu - i a_\mu) \chi_j + i A_\mu \bar{\chi} \gamma_\mu \tau^z \chi + \frac{i}{2\pi} a \wedge dB$$

$$\leftrightarrow \ \mathcal{L} = \sum_{j=1}^{2} \bar{\psi}_j \gamma_\mu (\partial_\mu - i b_\mu) \psi_j + i B_\mu \bar{\psi} \gamma_\mu \tau^z \psi + \frac{i}{2\pi} b \wedge dA$$

Comments:

3, there is also a SO(4) breaking fermion mass term, that drives the system into two different superfluid phases:

$$m\bar{\psi}\sigma^z\psi \sim -m\bar{\chi}\sigma^z\chi$$



## *Self-dual N=2 QED3*

4, evidences for the *N*=2 QED3 to be a CFT
4.1 Direct numerical evidence: Karthik, Narayanan arXiv:1606.04109

4.2 the tuning parameter $m\bar{\psi}\psi \sim -m\bar{\chi}\chi$ drives a topological phase transition, and the Chern-Simons level of the back ground field *A* and *B* change by +2 and -2 respectively. If we enhance *A* and *B* to SU(2) background gauge fields, their levels change by +1 and -1 respectively.

Simulation on a lattice model with the same transition (Slagle, You, Xu, arXiv:1409.7401, He, etc. arXiv:1508.06389),

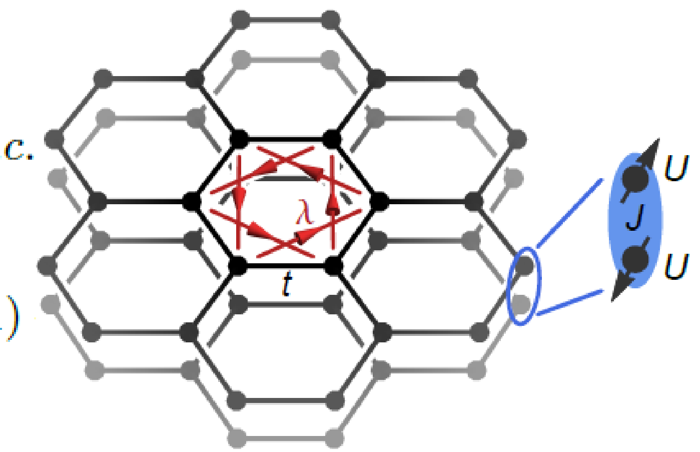
## *Sign problem free lattice model*

We want to design a similar lattice model with all the key physics, and "easy" to study numerically:

$$H = H_{\rm band} + H_{\rm int},$$

$$H_{\rm band} = -t \sum_{\langle ij \rangle, \ell} c_{i\ell}^\dagger c_{j\ell} + \sum_{\langle\langle ij \rangle\rangle, \ell} \mathrm{i}\lambda_{ij} c_{i\ell}^\dagger \sigma^z c_{j\ell} + H.c.$$

$$H_{\rm int} = +J \sum_i \left[ \boldsymbol{S}_{i1} \cdot \boldsymbol{S}_{i2} + \frac{1}{4}(n_{i1} - 1)(n_{i2} - 1) \right]$$




Simple limits of this model:

(1) Noninteracting: bilayer quantum spin Hall, boundary has two channels of gapless fermion modes

## *Sign problem free lattice model*

We want to design a similar lattice model with all the key physics, and "easy" to study numerically:

$$H = H_{\text{band}} + H_{\text{int}},$$

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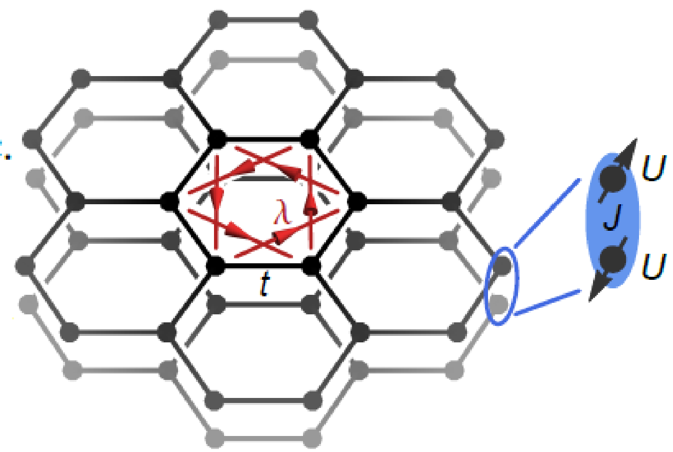
$$H_{\text{int}} = +J \sum_{i} \left[ \boldsymbol{S}_{i1} \cdot \boldsymbol{S}_{i2} + \frac{1}{4}(n_{i1}-1)(n_{i2}-1) \right]$$




Simple limits of this model:
(2) Strong *J*-interacting limit: trivial Mott insulator, with inter-layer spin singlet one every site. $|\Psi\rangle = \prod |\text{Singlet}\rangle$

What happens at intermediate *J* ?

## *Sign problem free lattice model*

Apparently, this model has at least $U(1)_{spin}$ x $U(1)_{charge}$ symmetry. At relatively weak interaction J, we can directly bosonize the edge states:

$$H_0 = \int dx \sum_{l=1}^{2} \psi_{l,L} i v \partial_x \psi_{l,L} - \psi_{l,R} i v \partial_x \psi_{l,R}$$

$$\Downarrow$$

$$H_0 = \int dx \sum_{l=1}^{2} \frac{v}{2K} (\partial_x \theta_l)^2 + \frac{vK}{2} (\partial_x \phi_l)^2$$

$\Downarrow$ interaction $H_v \sim \alpha \cos(2\pi\phi_1 - 2\pi\phi_2)$

$$\tilde{H} = \int dx \frac{\tilde{v}}{2\tilde{K}} (\partial_x \theta)^2 + \frac{\tilde{v}\tilde{K}}{2} (\partial_x \phi)^2$$

When $H_v$ is relevant, all the fermion modes are gapped at the boundary, but bosonic modes are gapless, and protected by symmetry. Thus the system becomes effectively a "bosonic topological insulator"

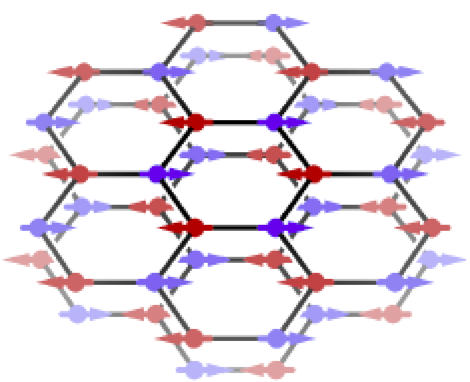
## *Sign problem free lattice model*

This model actually has an exact SO(4) symmetry. Spin-up and spin-down fermions have their individual SU(2) symmetry.
SO(4) vector:

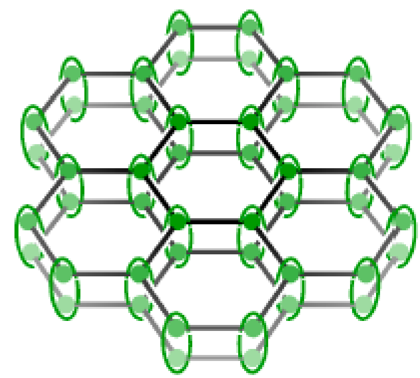
$$\begin{aligned} n_i &= (N_i^x, \mathrm{Im}\,\Delta_i, \mathrm{Re}\,\Delta_i, N_i^y). \\ &= f_{i\downarrow}^\dagger(\tau^0, \mathrm{i}\tau^1, \mathrm{i}\tau^2, \mathrm{i}\tau^3) f_{i\uparrow} + h.c., \end{aligned}$$

$$f_{i\uparrow} = \begin{pmatrix} c_{1i\uparrow} \\ (-1)^i c_{2i\uparrow}^\dagger \end{pmatrix}, f_{i\downarrow} = \begin{pmatrix} (-1)^i c_{1i\downarrow} \\ c_{2i\downarrow}^\dagger \end{pmatrix}$$

$(N^x, N^y)$



$\Delta$

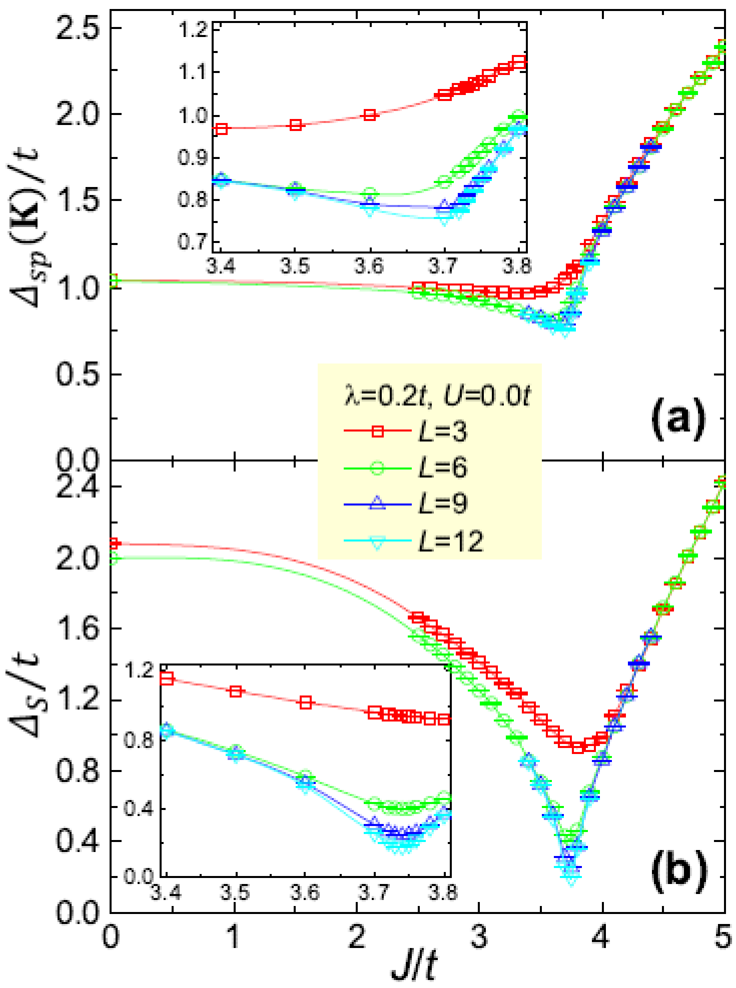


$$H = \sum_{i,j,\sigma} (-)^\sigma f_{i\sigma}^\dagger(-t_{ij} + i\lambda_{ij}) f_{j\sigma} + h.c. - \frac{J}{16} \sum_i \left(D_i D_i^\dagger + D_i^\dagger D_i\right) \qquad D_i = \sum_\sigma f_{i\sigma}\, i\tau^2\, f_{i\sigma}$$

## *Topological phase transition*

Determinant QMC data for bulk: (arXiv:1508.06389) we saw that the fermion gap is always finite, but bosonic modes, both spin and charge, becomes gapless at the SPT-trivial Quantum critical point. This is fundamentally different from free fermion topological transition.

Because the fermionic degrees of freedom never show up at low energy at either the boundary or the bulk quantum transition, the whole system can be viewed as a bosonic system.



## *Topological phase transition*

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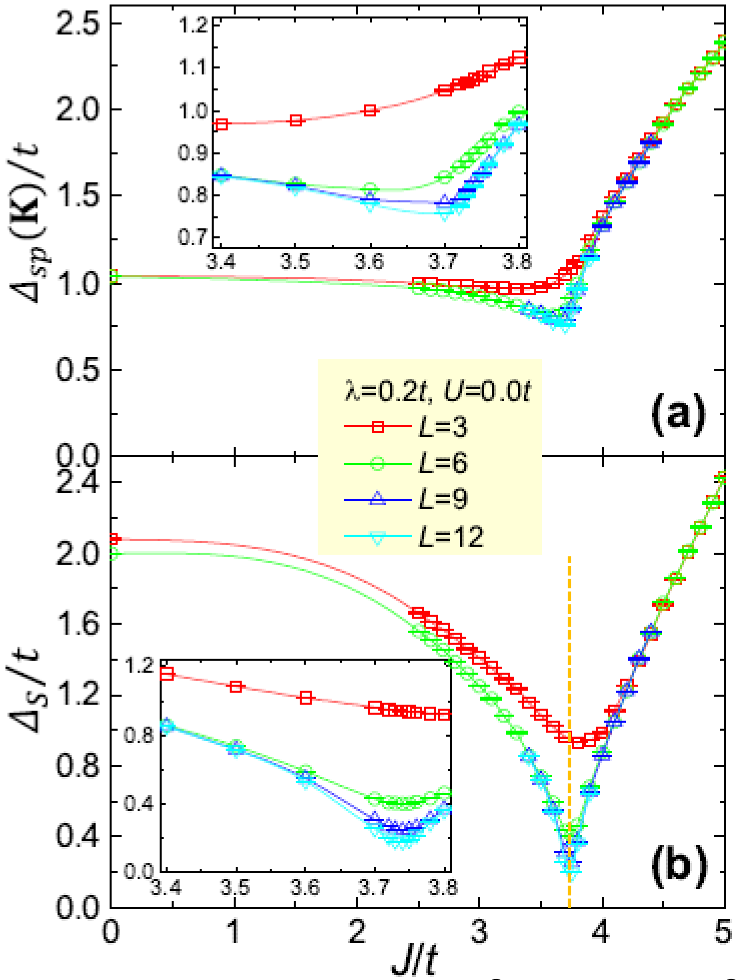
In terms of the N=2 QED language, this transition corresponds to changing the sign of the SU(2) invariant fermion mass term $m\bar{\psi}\psi$




$m<0$ $m>0$

## *More duality of N=2 QED3*

Another duality of the same theory (Potter, et.al. arXiv:1609.08618, Wang et.al. arXiv:1703.02426)

$$\mathcal{L} = \sum_{j=1}^{2} \bar{\chi}_j \gamma_\mu (\partial_\mu - i a_\mu) \chi_j + i A_\mu \bar{\chi} \gamma_\mu \tau^z \chi + \frac{i}{2\pi} a \wedge dB$$

Step 1: run the fermion-boson duality for each flavor:

$$\leftrightarrow \; \mathcal{L} = |(\partial_\mu - i b_\mu) z_1|^2 + g|z_1|^4 + |(\partial_\mu - i c_\mu) z_2|^2 + g|z_2|^4 + \frac{i}{2\pi} a \wedge (b - c) + \frac{i}{4\pi} b \wedge db - \frac{i}{4\pi} c \wedge dc$$

Step 2: Integrating out dynamical gauge field *a*:

$$\mathcal{L} = \sum_j |(\partial_\mu - i b_\mu) z_j|^2 + g|z_j|^4$$

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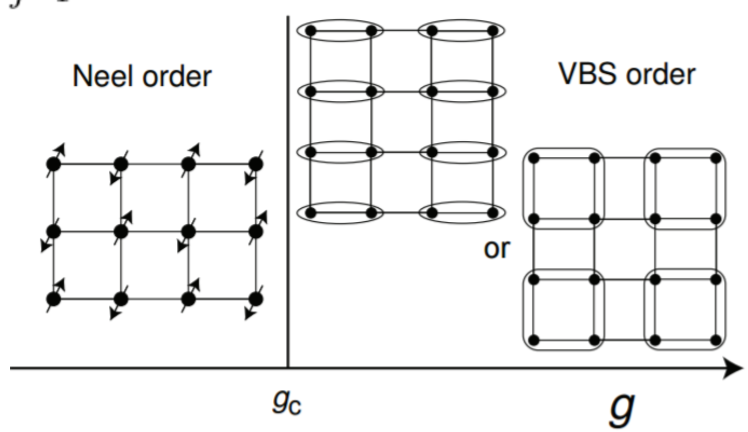
More duality of *N=2* QED3 (Potter, et.al. arXiv:1609.08618, Wang et.al. arXiv:1703.02426)

$$\mathcal{L} = \sum_j |(\partial_\mu - ib_\mu)z_j|^2 + r|z_j|^2 + g|z_j|^4$$

$r < 0$ SF ordered SF (VBS) order $r > 0$

$m < 0$ $m > 0$

$$\mathcal{L} = \sum_{j=1}^{2} \bar{\psi}_j \gamma_\mu (\partial_\mu - ia_\mu)\psi_j + m\bar{\psi}\sigma^z\psi$$



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More duality of *N=2* QED3 (Potter, et.al. arXiv:1609.08618, Wang et.al. arXiv:1703.02426)

$$\mathcal{L} = \sum_j |(\partial_\mu - ib_\mu) z_j|^2 + r|z_j|^2 + g|z_j|^4$$

$r < 0$ SF ordered

SF (VBS) order $r > 0$

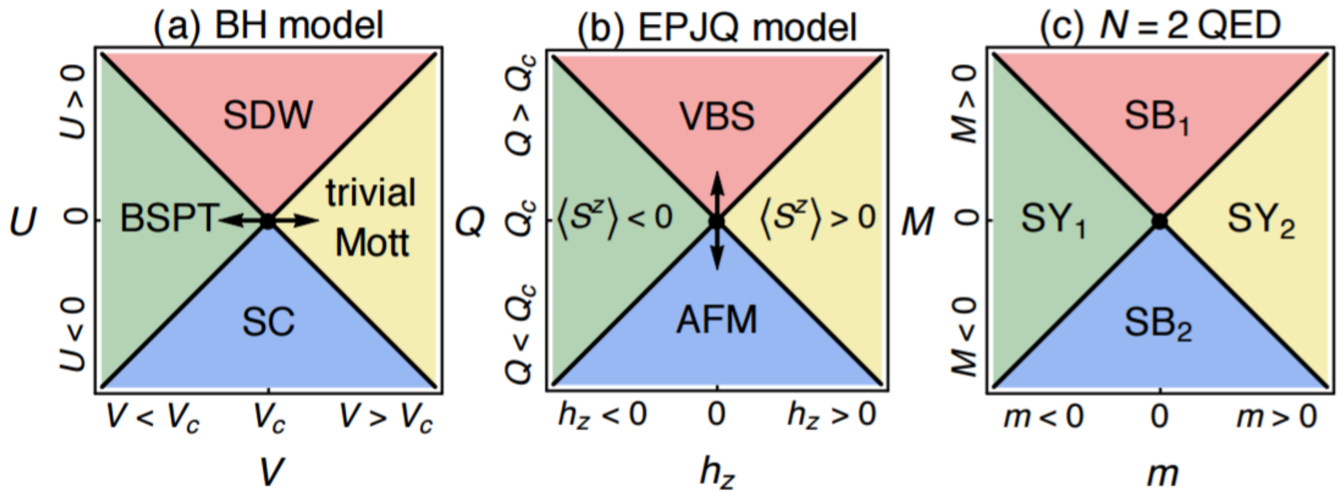
$m < 0$

$m > 0$

$$\mathcal{L} = \sum_{j=1}^{2} \bar{\psi}_j \gamma_\mu (\partial_\mu - i a_\mu) \psi_j + m \bar{\psi} \sigma^z \psi$$

All these are fancy theories…. What about predictions??

## *Implications of the proposed duality*




$$\xi \sim |V - V_c|^{-\nu_{\mathrm{BH}}},$$
$$\langle \rho_{i\uparrow}\rho_{i\downarrow}\rho_{j\uparrow}\rho_{j\downarrow}\rangle \sim |\boldsymbol{r}_{ij}|^{-1-\eta^{\rho}_{\mathrm{BH}}},$$
$$\langle \Delta_i^\dagger \Delta_j\rangle \sim |\boldsymbol{r}_{ij}|^{-1-\eta^{\Delta}_{\mathrm{BH}}};$$

$$\xi \sim |Q - Q_c|^{-\nu^{xy}_{\mathrm{JQ}}},$$
$$\langle S_i^z S_j^z\rangle \sim |\boldsymbol{r}_{ij}|^{-1-\eta^{z}_{\mathrm{JQ}}},$$
$$\langle S_i^+ S_j^-\rangle \sim |\boldsymbol{r}_{ij}|^{-1-\eta^{xy}_{\mathrm{JQ}}}.$$

Duality predicts:
Wang et.al. arXiv:1703.02426

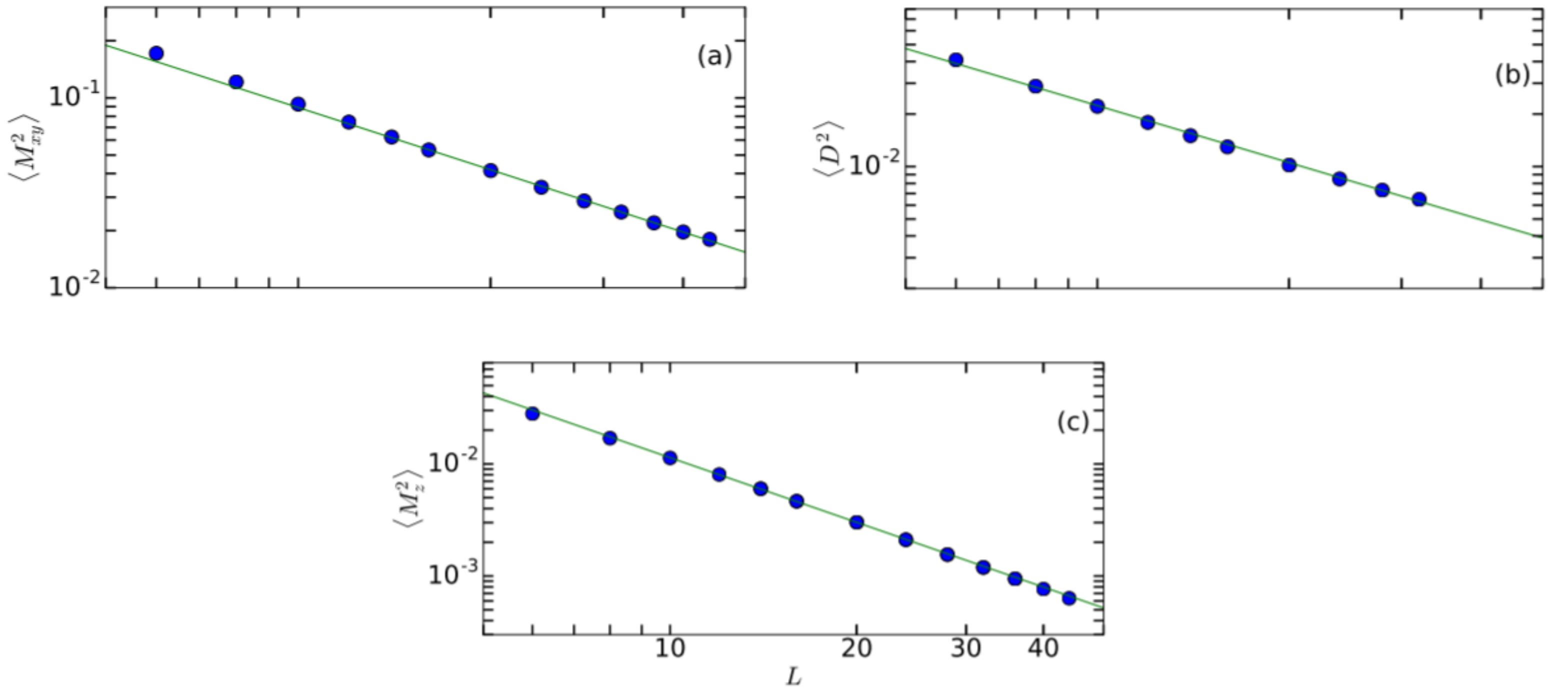
$$\boxed{\begin{aligned} &3 - \frac{1}{\nu_{\mathrm{BH}}} = \frac{1+\eta^{z}_{\mathrm{JQ}}}{2},\\ &3 - \frac{1}{\nu^{xy}_{\mathrm{JQ}}} = \frac{1+\eta_{\mathrm{QED}}}{2} = \frac{1+\eta^{\rho}_{\mathrm{BH}}}{2},\\ &\eta^{\Delta}_{\mathrm{BH}} = \eta^{xy}_{\mathrm{JQ}}.\end{aligned}}$$

## *The easy-plane J-Q model with a continuous AF-VBS transition*

$$H_{\mathrm{JQ}} = -J \sum_{\langle ij \rangle} (P_{ij} - \Delta S_i^z S_j^z) - Q \sum_{\langle ijklmn \rangle} P_{ij} P_{kl} P_{mn},$$

$$P_{ij} = \tfrac{1}{4} - \boldsymbol{S}_i \cdot \boldsymbol{S}_j,$$

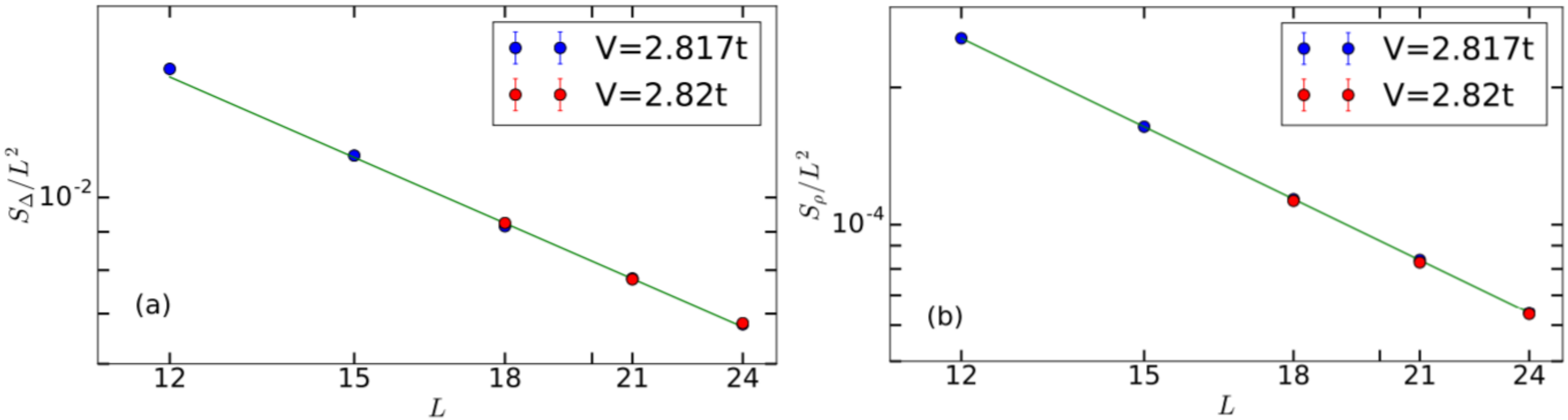
We look at $\Delta = ½$ : $\eta_{AF} = \eta_{VBS} = \eta_{\mathbf{JQ}}{}^{\mathbf{xy}} \sim 0.13(3)$, $\eta_{\mathbf{JQ}}{}^{\mathbf{z}} \sim 0.91(3)$



## *Further analysis of the Bilayer-Honeycomb (BH) model*

$$H_{\rm band} = -\,t\sum_{\langle ij\rangle,\ell} c^\dagger_{i\ell}c_{j\ell} + \sum_{\langle\langle ij\rangle\rangle,\ell} \mathrm{i}\lambda_{ij}c^\dagger_{i\ell}\sigma^z c_{j\ell} + H.c.,$$

$$H_{\rm int} = \;\; +J\sum_i \left[\boldsymbol{S}_{i1}\cdot\boldsymbol{S}_{i2} + \frac{1}{4}(n_{i1}-1)(n_{i2}-1) - \frac{1}{4}\right]$$



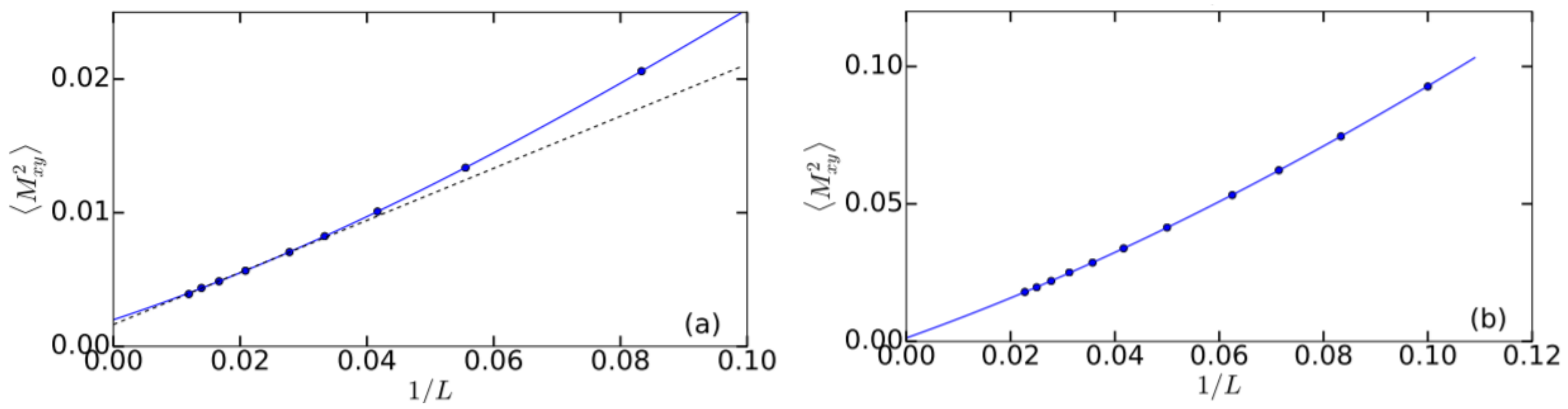

$\eta_{\rm BH}{}^{\Delta} \sim 0.10(1)$, $\eta_{\rm BH}{}^{\rho} \sim 1.00(1)$ (consistent with lattice QED numerics)
More data available at arXiv:1705.10670. Within error bar, these two quantum critical points do seem dual to each other!!

## *The easy-plane J-Q model with a continuous AF-VBS transition*

$$H_{\mathrm{JQ}} = -J \sum_{\langle ij \rangle} (P_{ij} - \Delta S_i^z S_j^z) - Q \sum_{\langle ijklmn \rangle} P_{ij} P_{kl} P_{mn},$$

$$P_{ij} = \tfrac{1}{4} - \boldsymbol{S}_i \cdot \boldsymbol{S}_j,$$

Compare Δ = 1, and Δ = ½:



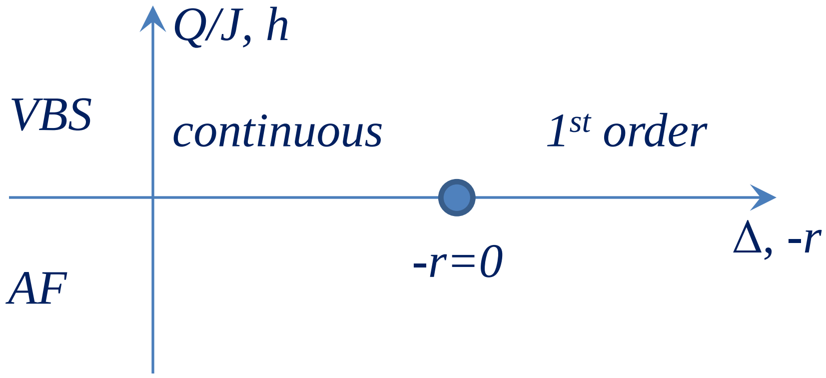

Tri-critical point by tuning Δ?

## *The easy-plane J-Q model with a continuous AF-VBS transition*

$$H_{\mathrm{JQ}} = -J\sum_{\langle ij\rangle}(P_{ij} + \Delta S_i^z S_j^z) - Q\sum_{\langle ijklmn\rangle} P_{ij}P_{kl}P_{mn}$$

$$P_{ij} = \tfrac{1}{4} - \boldsymbol{S}_i \cdot \boldsymbol{S}_j,$$

Tri-critical point by tuning Δ?




Proposed theory:

$$\sum_{j=1}^{2}\bar{\psi}_j(\partial_\mu - ia_\mu)\psi_j + v\bar{\psi}\sigma^3\psi\phi + (\partial_\mu\phi)^2 + r\phi^2 + g\phi^4 + h\phi$$

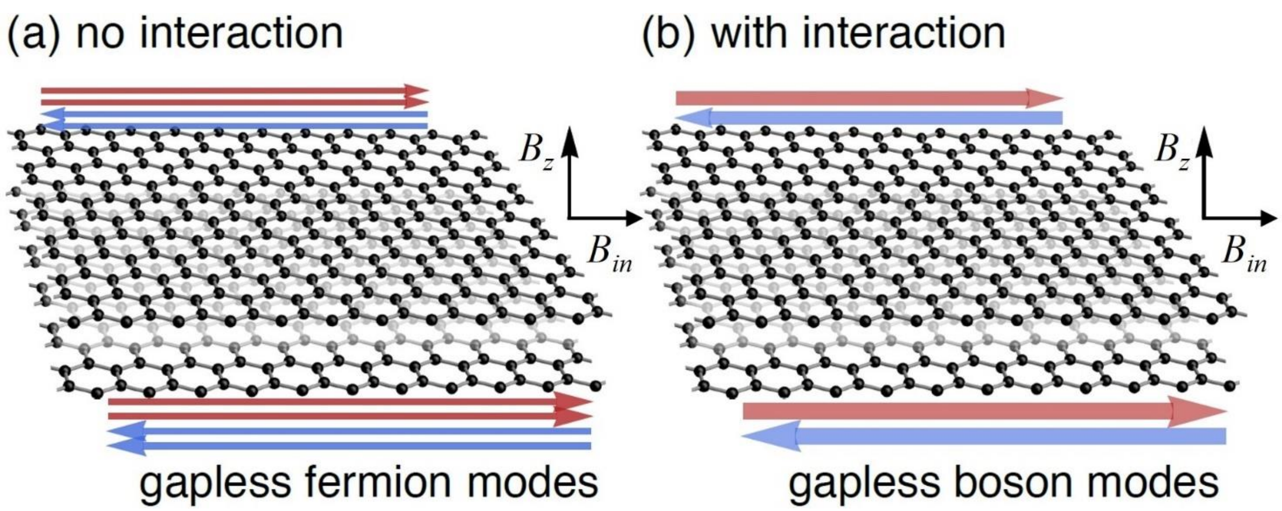
## *Potential Experimental Platform of Bosonic SPT, and Topological Transition*

**Claim:**

Bilayer graphene under (strong) magnetic field (with both $z$ and inplane components), will be driven into a "bosonic" SPT state with U(1)xU(1) symmetry by Coulomb interaction.

**Meaning:**

boundary states must remain gapless with U(1)xU(1) symmetry, but, only protected gapless bosonic modes, no gapless fermion modes, under Coulomb interaction. (Bi, et.al. arXiv:1602.03190)

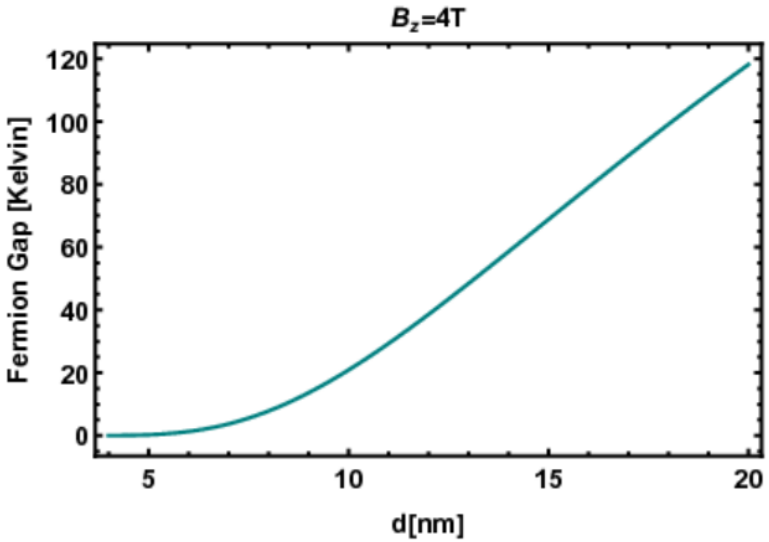


## *Potential Experimental Platform of Bosonic SPT, and Topological Transition*

**Predictions:**

Main prediction, the boundary of bilayer graphene under B field is a conductor with a single particle gap;

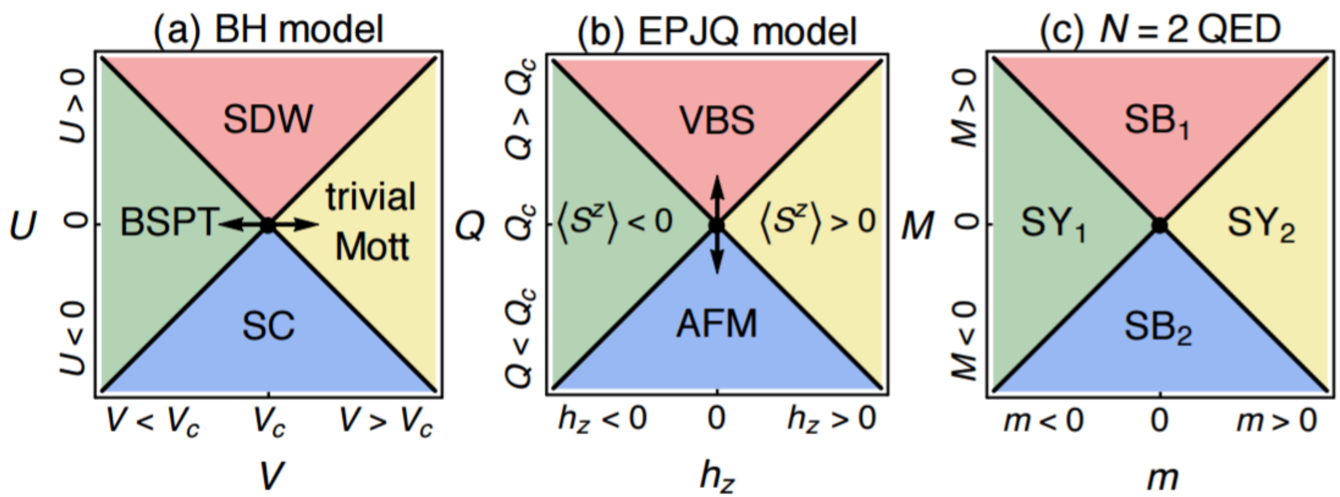
Single particle gap comes from interaction, which is tunable by tuning the distance to the metallic gate (screening):




Tunneling from a normal tip would see this gap, but a superconductor tip would see zero gap.

## *Summary*

Proposed duality between two exotic quantum critical points: bosonic topological transition and the easy-plane deconfined quantum critical point;




Numerical simulation does support the theoretical predictions!