Nonequilibrium Dynamics in Integrable Quantum Many-Body Systems

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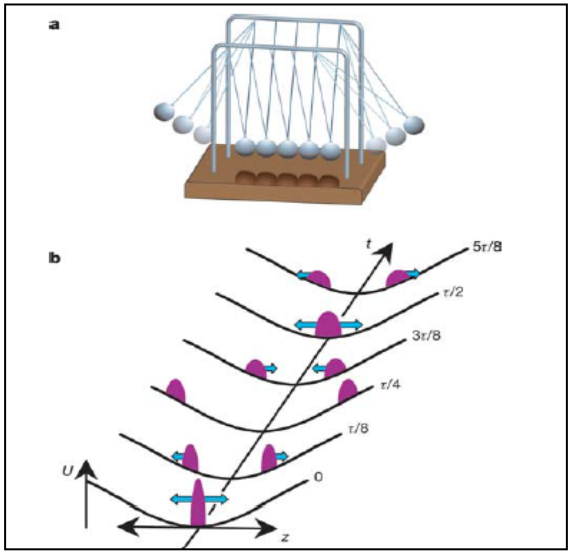
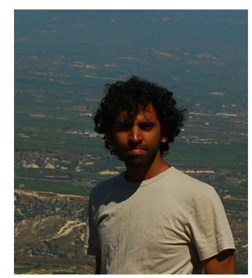
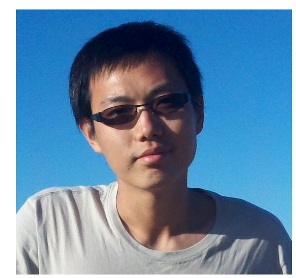


Kinoshita, Wenger, Weiss (Nature '06)

Garry Goldstein



Deepak Iyer



Wenshuo Liu

Congratulations, KITS!

New Horizons in Condensed Matter Physics - KITS, Beijing (March, 2017)

Time Evolution of systems out of equilibrium

- **Prepare an isolated quantum many-body system in state $|\Phi_0\rangle$,** typically eigenstate of H_0
- **At , $t=0$ evolve system with $H(t)$:**

If the Hamiltonian is time independent,

$$|\Phi_0, t\rangle = e^{-iHt}|\Phi_0\rangle$$

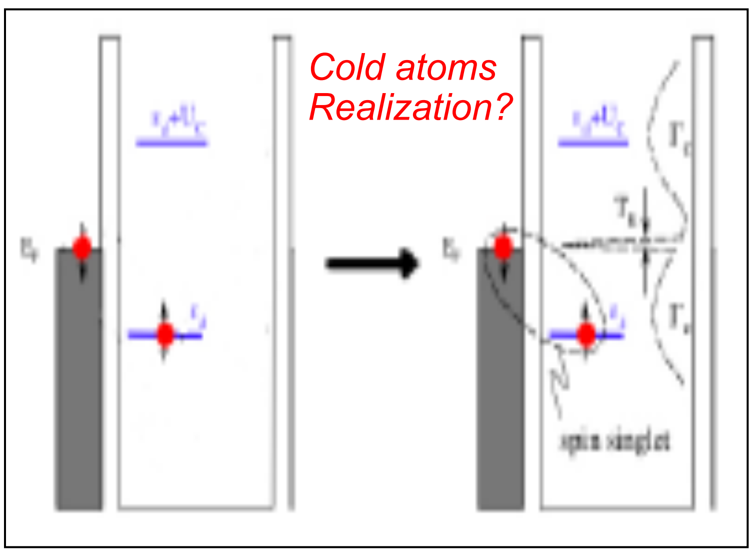
If the Hamiltonian is time dependent,

$$|\Phi_0, t\rangle = Te^{-i\int_0^t H(t')dt'}|\Phi_0\rangle$$

- **Many experiments: cold atom systems, nano-devices, molecular electronics, photonics :** *new systems, old questions*

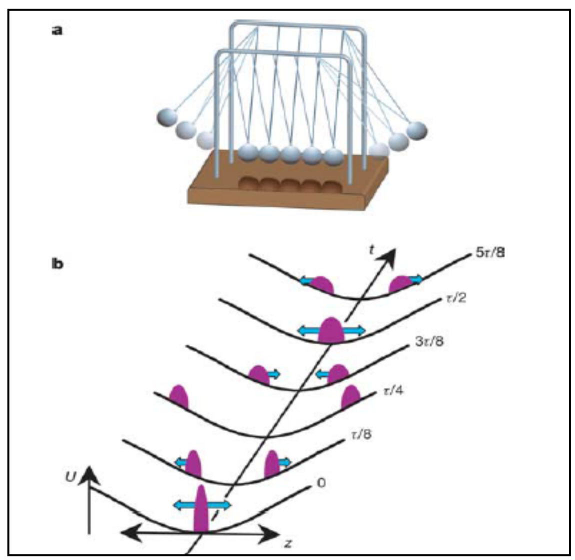
Time evolution of observables: $\langle O(t)\rangle = \langle \Phi_0, t|O|\Phi_0, t\rangle$

- **Manifestation of interactions**



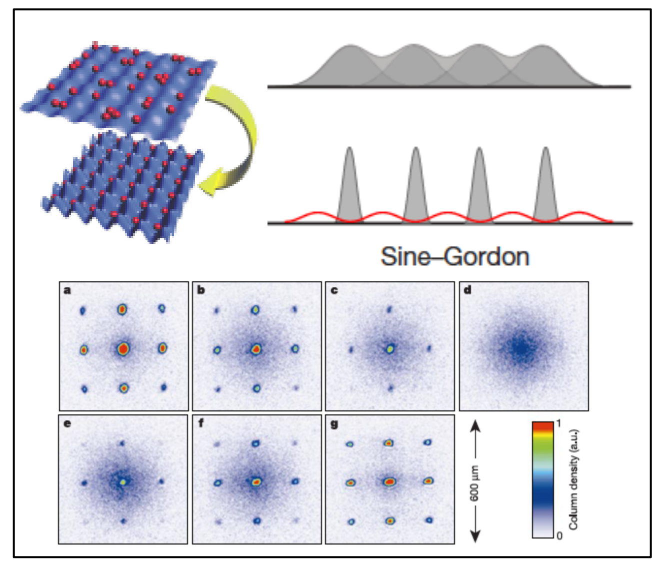
Time evolution of the Kondo peak.
- Time resolved photo emission spectroscopy

Kinoshita et al '06



Newton's Cradle

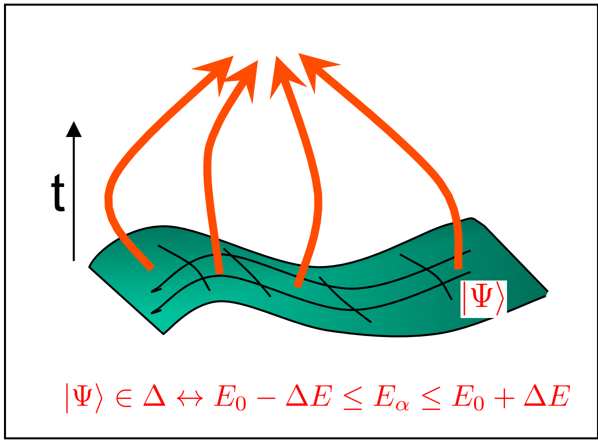
Greiner et al '02 *Bloch et al '08*



Mott insulator ←→ superfluid :2d, 1d

Quenching – long time limit, thermalization

Time evolution and statistical mechanics: equilibrium



$$\langle A(t)\rangle = \langle \Phi_0|e^{iHt} A e^{-iHt}|\Phi_0\rangle \underset{t\to\infty}{\longrightarrow} \bar{A}_{\Phi_0}$$

- **Long time limit and thermalization:**

- is there a density operator ρ such that $\bar{A} = Tr(\rho A)$?
- does it depend only on $E_0 = \langle \Phi_0|H|\Phi_0\rangle$, not on $|\Phi_0\rangle$?

$$\langle A(t)\rangle = \sum_{\alpha,\beta} \langle \Phi_0|\alpha\rangle A_{\alpha\beta} \langle \beta|\Phi_0\rangle e^{i(E_\alpha - E_\beta)t} \underset{t\to\infty}{\rightarrow} \sum_\alpha |\langle \Phi_0|\alpha\rangle|^2 A_{\alpha\alpha}$$

$$\stackrel{?}{=} \langle A\rangle_{\text{microcan}}(E_0) \equiv \frac{1}{N_{E_0,\Delta E}} \sum_{\alpha\in\Delta E} A_{\alpha\alpha}$$

- **Gibbs Ensemble**

- Non integrable models: ETH $A_{\alpha\alpha} = \langle \alpha|A|\alpha\rangle = f_A(E_\alpha)$, with f smooth function of E_α

Deutsch '92, Srednicki '94

→ Gibbs ensemble: $\langle A(t\to\infty)\rangle = \frac{1}{Z} Tr(e^{-\beta H} A)$, with β det. by $\langle H\rangle_{t=0} = \frac{1}{Z} Tr(e^{-\beta H} H)$

- **Generalized Gibbs ensemble (GGE)**

- Integrable models: local conserved charges, I_n, GETH $\langle \alpha|A|\alpha\rangle = f(I_{1,\alpha}, I_{2,\alpha}, \ldots)$

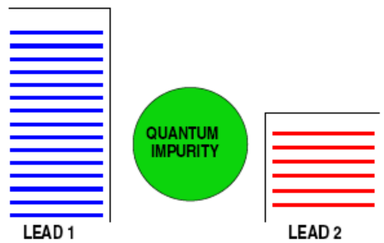
$$\langle A(t\to\infty)\rangle = Tr(\hat{\rho}A) \quad \hat{\rho} = Z^{-1}\exp(-\sum \beta_n I_n) \quad \textit{with} \quad \langle I_n\rangle_{t=0} = Tr(I_n\hat{\rho})$$ *Rigol et al*

Quenching and non-thermalization (no MBL)

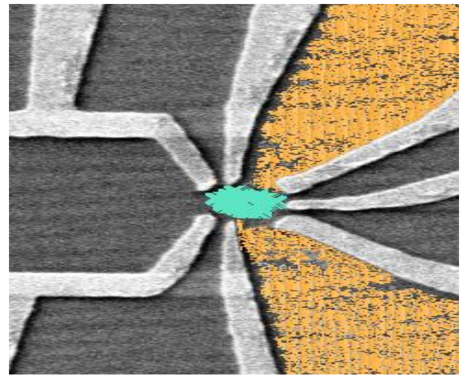
Nonequilibrium currents

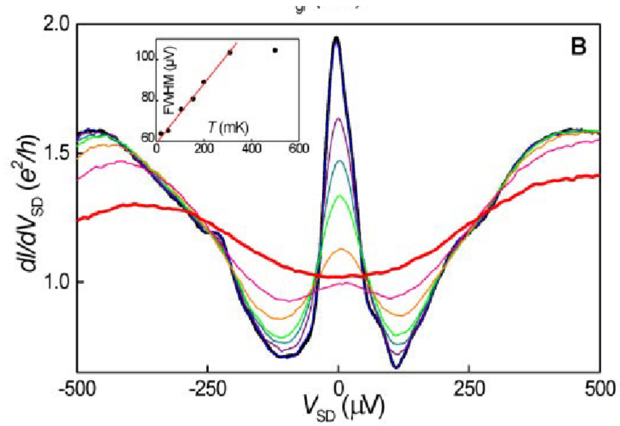
Goldhaber-Gordon *et al*, Cronenwett *et al*, Schmid *et al*

- **Two baths or more:**
  - time evolution in a nonequilibrium set up:



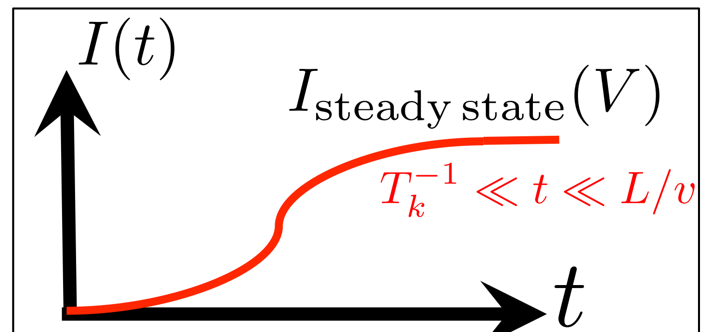
$H = H_{leads} + H_{imp} + H_{leads-imp}$





- $t \leq 0$, leads decoupled, system described by: ρ_o
- $t = 0$, couple leads to impurity
- $t \geq 0$, evolve with $H(t) = H_0 + H_1$

Interplay - strong correlations and nonequilibrium

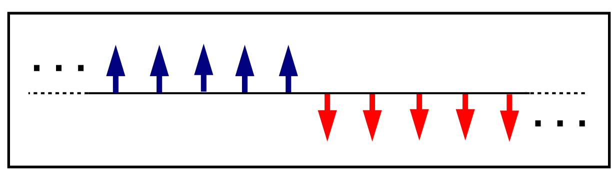


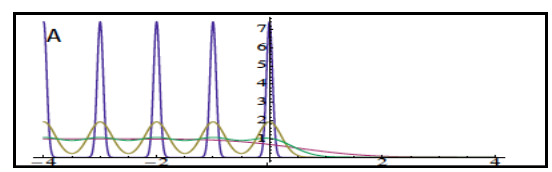
- **What is the time evolution of the current** $\langle I(t) \rangle$ **?**

- **Long time limit: Under what conditions is there a nonequilibrium steady state (NESS)? Dissipation mechanism?**

- **Steady state – is there a non-thermal** ρ_{NESS} **? Voltage dependence?**
- **New effects out of equilibrium? New scales? Phase transitions, universality?**

- **Domain wall:** spin currents, NESS
  $t \to \infty, L \to \infty$





- ***Newton's Cradle*** (no NESS)

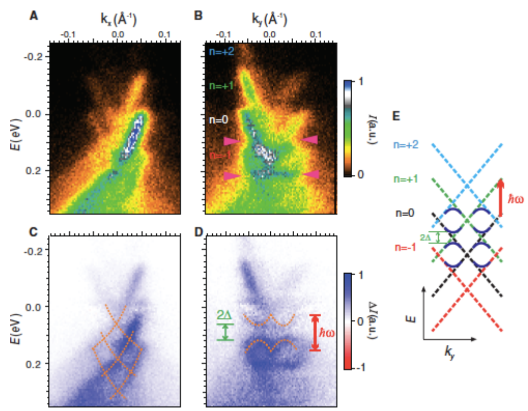
(Periodically) Driven systems

- A *Floquet* Hamiltonian $H(t) = T(t+T)$ has solutions of the form:

$$\psi_\alpha(x,t) = e^{-i\epsilon_\alpha t}\phi_\alpha(x,t)$$

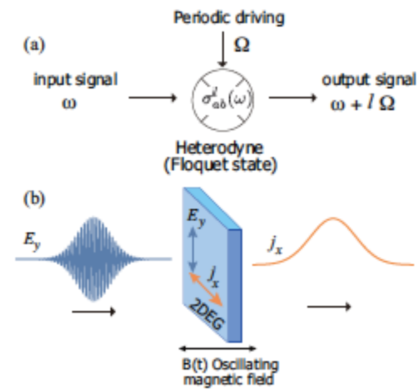
with $\phi_\alpha(x,t)$ periodic and the *quasi energies* ϵ_α determined up to $\omega = 2\pi/T$

- Many experimental realizations: pump-probe, state engineering, photoinduced Floquet- Weyl semimetal phases..



Periodically driven TI:
Floquet – Bloch states

Gedik et al '13



Heterodyne Hall Efect

Oka and Bucciantini '16

- Driven Lieb-Liniger (in a box)

$$H = -\int b^\dagger(x)\partial^2 b(x)\,dx + c\int b^\dagger(x)b(x)b^\dagger(x)b(x)\,dx + f(t)\int x\,b^\dagger(x)b(x)\,dx$$

Floquet spectrum, heating, synchronicity, turbulence..

Outline

Wish to study these questions exactly and confront them with experiment

1. Quench evolution in quantum integrable many-body systems

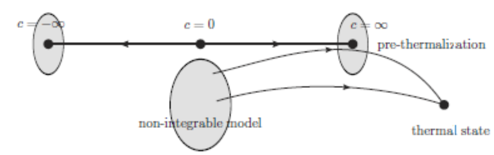
   - *Yudson's contour approach (infinite volume system): (Yudson '85)* $L = \infty, N$ fixed

2. Bosons on the continuous line with short range interactions *(Lieb-Liniger model)*

   A. Finite boson system: $N - \text{finite}, L \to \infty,\ t \ll L/v_{typ}$

   Hanbury Brown –Twiss effect and RG flow in time

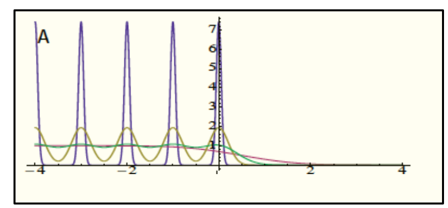
   Floquet driven system



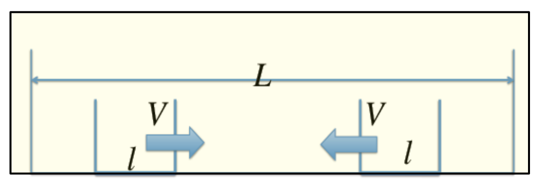
   B. Thermodynamic boson system $N, L \to \infty,\ n = N/L \text{ fixed},\ t \ll L/v_{typ}$

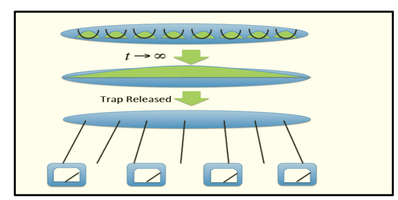
   - Generalizing Yudsons' approach to thermodynamic systems

     Time evolution of observables – monster formula

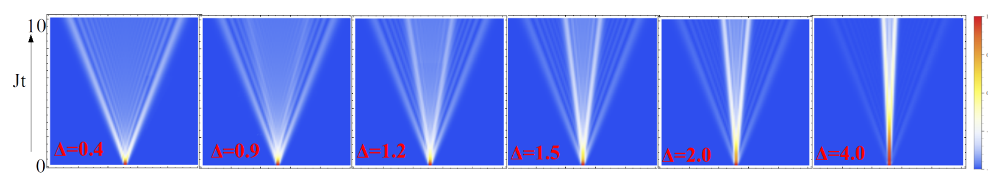


   - Non equilibration (NESS): the Domain Wall
   - Equilibration and GGE for repulsive interaction:
     - Quench from Mott to Lieb-Liniger fluid
     - Newton's Cradle on the average (poor man's version)





3. *Quenching the XXZ Heisenberg model*



Quenching in 1-d systems

Physical Motivation:

- **Natural dimensionality of many systems:**

  - wires, waveguides, optical traps, edges

- **Impurities: Dynamics dominated by s-waves, reduces to 1D system**
- **Many experimental realizations: ultracold atom traps, nano-systems..**

Special features of 1- d : theoretical

- **Strong quantum fluctuations for any coupling strength**
- **Powerful mathematical methods:**

  - RG methods, Bosonization, CFT methods, Bethe Ansatz approach

- Bethe Ansatz approach: allows complete diagonalization of *H*

- Experimentally realizable: *Hubbard model, Kondo model, Anderson model, Lieb-Liniger model, Sine-Gordon model, Heisenberg model, Richardson model..*

- BA $\longrightarrow$ Quench dynamics of many body systems? Exact!

Others approaches: Keldysh, t-DMRG, t-NRG, t-RG, exact diagonalization..

Time Evolution and the Bethe Ansatz

- A given state $|\Phi_0\rangle$ can be formally time evolved in terms of a complete set of energy eigenstates $|F^\lambda\rangle$

$$|\Phi_0\rangle = \sum_\lambda |F^\lambda\rangle\langle F^\lambda|\Phi_0\rangle \longrightarrow |\Phi_0, t\rangle = e^{-iHt}|\Phi_0\rangle = \sum_\lambda e^{-i\epsilon_\lambda t}|F^\lambda\rangle\langle F^\lambda|\Phi_0\rangle$$

If H integrable $\longrightarrow$ eigenstates $|F^\lambda\rangle$ are known via the Bethe-Ansatz

- Use *Bethe Ansatz* to study quench evolution and nonequilibrium

- New technology is necessary:

- **Standard approach**: **PBC** $\longrightarrow$ **Bethe Ansatz eqns** $\longrightarrow$ **spectrum** $\longrightarrow$ **thermodynamics**
- **Non equilibrium entails *additional* difficulties:**

i. Compute overlaps (form factors) ii. Compute matrix elements iii. Sum over complete basis

J. S. Caux et al, Essler et al, Calabrese et al

- Floquet system: expand initial state $|\Phi_0\rangle$ in terms of *Floquet* states

$$|\Phi_0\rangle = \sum c_\alpha|\psi_\alpha\rangle \quad \text{then} \quad \mathcal{F}^n|\Phi_0\rangle = \sum c_\alpha e^{i\epsilon_\alpha nT}|\psi_\alpha\rangle$$

Yudson's contour representation (infinite volume)

Instead of $|\Phi_0\rangle = \sum_\lambda |F^\lambda\rangle\langle F^\lambda|\Phi_0\rangle$ introduce (directly in infinite volume):

Contour representation of $|\Phi_0\rangle$

$$|\Phi_0\rangle = \int_\gamma d^N\lambda \, |F^\lambda\rangle(F^\lambda|\Phi_0\rangle$$

V. Yudson, sov. phys. *JETP* (1985)

Computed S-matrix of Dicke model

with: $|F^\lambda\rangle$ Bethe eigenstate

$|F^\lambda)$ obtained from Bethe eigenstate by setting $S = I$ - One quadrant suffices

γ contour in momentum space $\{\lambda\}$ determined by **pole structure** of $S(\lambda_i - \lambda_j)$

Note: in the infinite volume limit momenta $\{\lambda\}$ are not quantized
- no Bethe Ansatz equations, $\{\lambda\}$ free parameters

then:

$$|\Phi_0, t\rangle = \int_\gamma d^N\lambda \, e^{-iE(\lambda)t}|F^\lambda\rangle(F^\lambda|\Phi_0\rangle$$

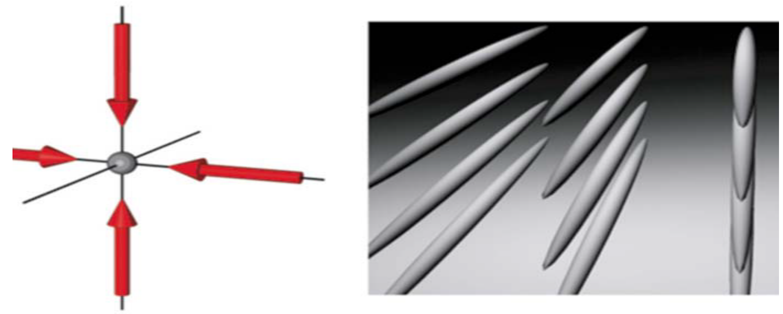
- Describes systems in the zero density limit
- Generalize to thermodynamic systems with finite density

Ultracold Atoms – the Lieb Liniger model

Gas of neutral atoms moving on the line and interacting with short-range interaction

Short range interaction among atoms: $V(x_1 - x_2) = c\delta(x_1 - x_2)$

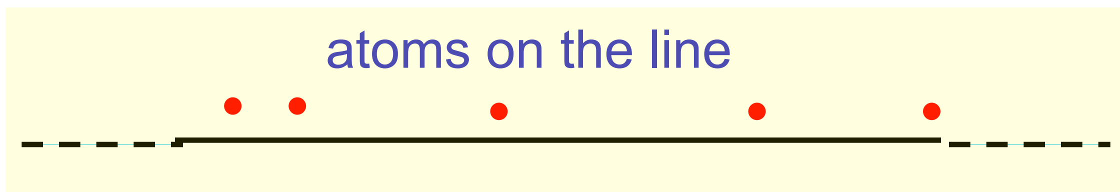
$$H_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j<l} \delta(x_j - x_l)$$



Bloch et al '08

$c > 0$ *repulsive*
$c < 0$ *attractive*

Can be tune by Feshbach resonance



Comment:

- Very short range interaction. Valid for low densities,

$$l = L/N \gg l_{\text{Van der Vaals}}$$

- The description of physics depends on the scale of observation

Bosonic system – BA solution

The N-boson eigenstatestate (Lieb-Liniger '67)

$$|\lambda_1,\cdots,\lambda_N\rangle = \int_y \prod_{i<j} Z^y_{ij}(\lambda_i-\lambda_j) \prod_j e^{i\lambda_j y_j}\; b^\dagger(y_j)|0\rangle$$

- **Eigenstates labeled by Momenta** $\lambda_1,\cdots,\lambda_N$

- ***Thermodynamics***: *impose PBC* $\longrightarrow$ *BA eqns* $\longrightarrow$ *momenta* $\begin{cases} \textit{real} & c>0 \\ \textit{n-strings} & c<0 \end{cases}$ ⁝

- ***Dynamics*** *(infinite volume): momenta unconstrained*

- **Dynamic factor:** $Z^y_{ij}(\lambda_i-\lambda_j) = \dfrac{\lambda_i-\lambda_j-ic\,sgn(y_i-y_j)}{\lambda_i-\lambda_j-ic} = \begin{cases} 1 & y_i>y_j \\ S^{ij} & y_i<y_j \end{cases}$

- *The 2-particle S-matrix:* $S_{ij}(\lambda_i-\lambda_j) = \dfrac{\lambda_i-\lambda_j+ic}{\lambda_i-\lambda_j-ic}$ *enters when the particles cross*

- poles of the S-matrix at: $\lambda_i = \lambda_j + ic$

- **The energy eigenvalues**

$$H|\lambda_1,\cdots,\lambda_N\rangle = \sum_j \lambda_j^2\;|\lambda_1,\cdots,\lambda_N\rangle$$

bosonic system: contour representation

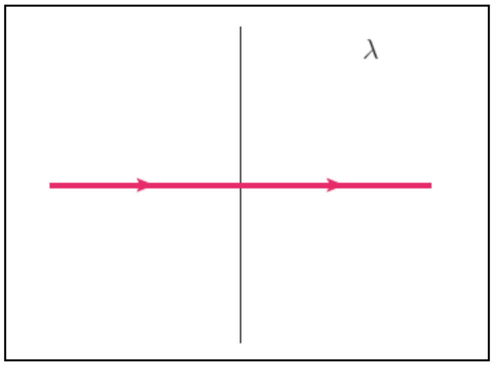
"Central theorem"

denote:

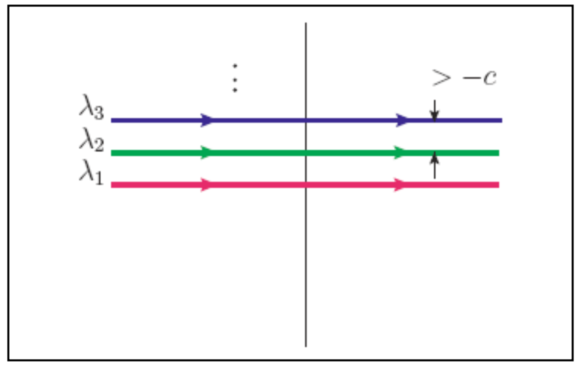
$$\theta(\vec{x}) = \theta(x_1 > x_2 > \cdots .x_N)$$

$$|\Phi_0\rangle = \int_x \Phi_0(\vec{x}) b^\dagger(x_N)\cdots b^\dagger(x_1)|0\rangle =$$

$$= \int_{x,y}\int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} \prod_j e^{i\lambda_j(y_j - x_j)} b^\dagger(y_j)|0\rangle$$



Repulsive c > 0



Attractive c < 0,

contour accounts for strings, bound states

It time evolves to:

$$|\Phi_0, t\rangle = \int_x \int_y \int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} \prod_j e^{-i\lambda_j^2 t} e^{i\lambda_j(y_j - x_j)} b^\dagger(y_j)|0\rangle$$

- **Expression contains full information about the dynamics of the system**

What to calculate?

- **We shall study:**

1. Evolution of the density $\rho(x) = b^\dagger(x) b(x)$

$$C_1(x,t) = \langle \Phi_0, t | b^\dagger(x) b(x) | \Phi_0, t \rangle = \int dx_1 .. dx_N \, |\Phi_0(x_1, .., x_N, t)|^2 \sum_j \delta(x - x_j)$$

- The probability to find the bosons at point x at time t
if at time $t = 0$ they started with wave function $\Phi_0(x_1, .., x_N)$

- Can be measured by Time of Flight experiments

- **competition** between quantum broadening and attraction

2. Evolution of noise correlation

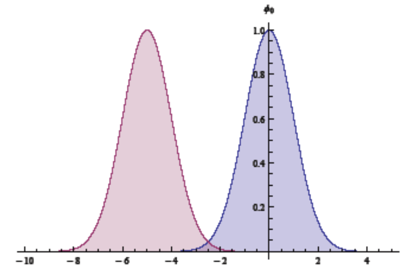
$$C_2(x,y,t) = \frac{\langle \Phi_0, t | \rho(x) \rho(y) | \Phi_0, t \rangle}{C_1(x,t) C_1(y,t)} - 1$$

- Time dependent Hanbury Brown - Twiss effect

Evolution of a bosonic system: density

- Consider an initial state:

$$\Phi_0(x_1,x_2) = \frac{1}{(\pi\sigma^2)^{\frac{1}{4}}}\, e^{-\frac{(x_1)^2+(x_2+a)^2}{2\sigma^2}}$$

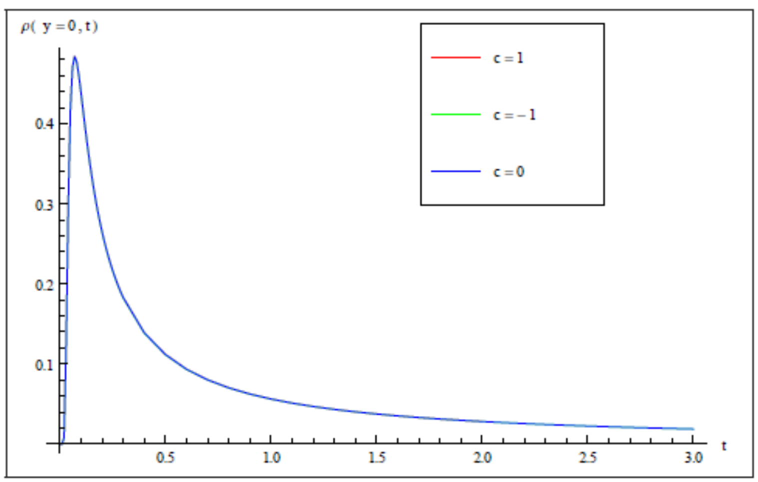


- Its evolution is:

$$\Phi_0(y_1,y_2,t) = \int_x \Phi_0(x_1,x_2)\frac{1}{4\pi i t}\, e^{i\frac{(y_1-x_1)^2}{4t}+i\frac{(y_2-x_2)^2}{4t}}\left[1 - c\sqrt{\pi i t}\,\theta(y_2-y_1)\, e^{\frac{i}{8t}\alpha^2(x,y,t)}\mathrm{erfc}\left(\frac{i-1}{4}\frac{i\alpha(x,y,t)}{\sqrt{t}}\right)\right]$$

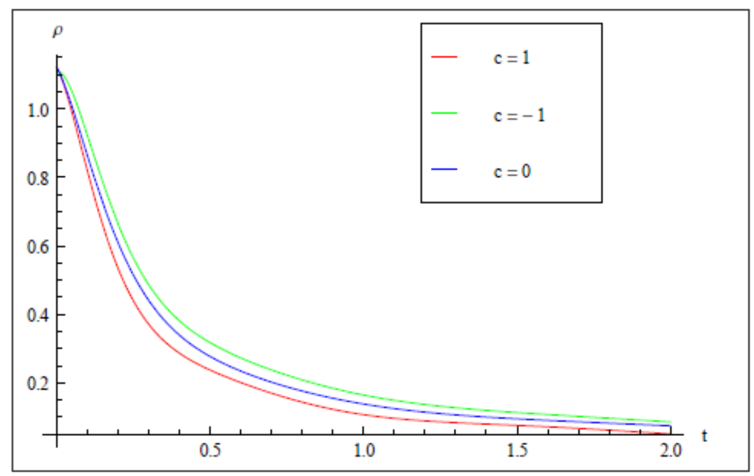
- Compute the evolution of the density $\rho(0,t)$:

***i.* Initial condition:** $a \gg \sigma$



No interaction effects- small initial overlap, then density too low

***ii.* Initial condition:** $a \ll \sigma$

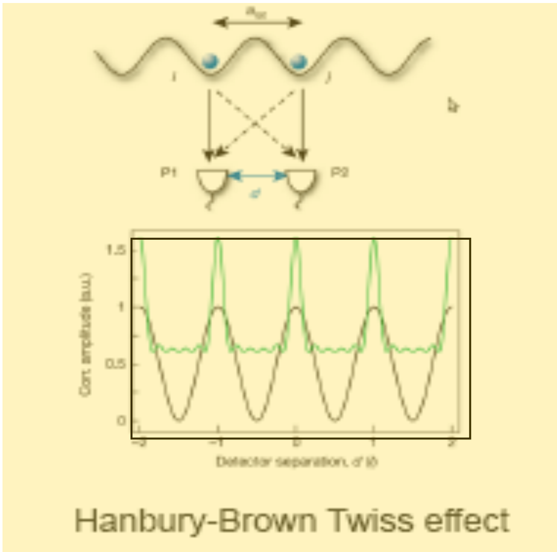


Strong interaction effects: initial overlap

The Hanbury Brown – Twiss effect

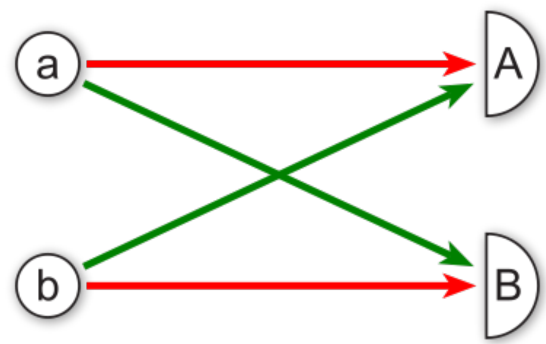
- Hanbury brown-Twiss effect for two bosons

Bloch et al.
RMP ‘08



Measure: $C_2(x_1, x_2, t)$

- Two stars: a, b



Free bosons $C_2(x, -x) \sim \cos x$

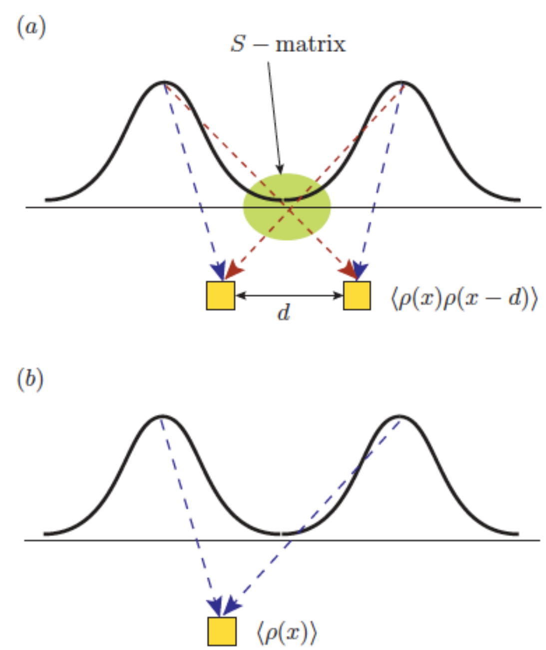
Free Fermions $C_2(x, -x) \sim -\cos x$

Time dependent Hanbury Brown - Twiss effect

- Time dependent
- Many bosons
- More structure: main peaks, sub peaks
- Effects of interactions?

▪ repulsive bosons evolve into fermions

• attractive bosons evolve to a condensate



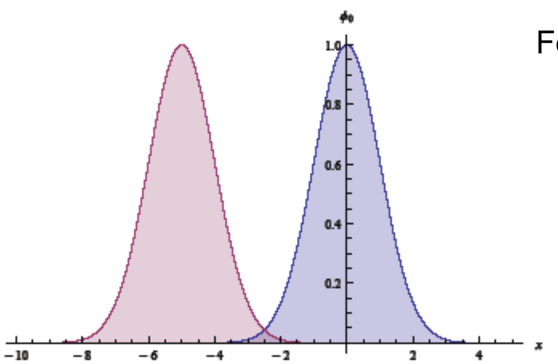
Evolution of repulsive bosons into fermions: HBT

Density - Density correlation: *long time asymptotics - repulsive:* *Iyer, NA '13*

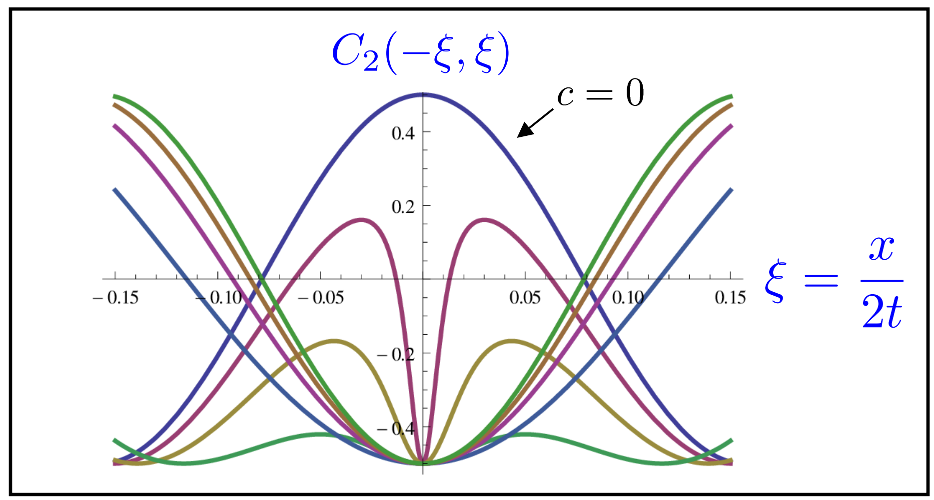
- **Bosons turn into fermions as time evolves (for any $c > 0$)**
- **Can be observed in the noise correlations: (dependence on t only via $\xi = x/2t$)**

$$C_2(x_1, x_2, t) \to C_2(\xi_1, \xi_2) = \frac{\langle \rho(\xi_1)\rho(\xi_2) \rangle}{\langle \rho(\xi_1) \rangle \langle \rho(\xi_2) \rangle} - 1,$$

$c/a = 0, .3, \cdots, 4$



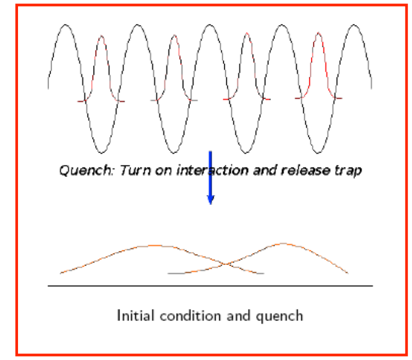
Fermionic correlations evolve



- **Fermionic dip develops for any repulsive interaction on time scale set by $1/c^2$**

Evolution of a bosonic system: noise correlations

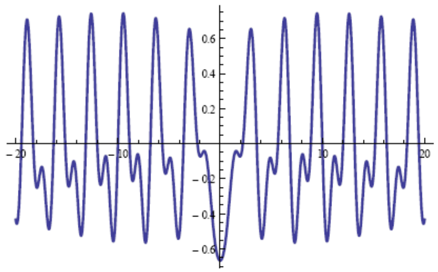
Noise correlations – starting from a lattice



Repulsive bosons

$C_2(\xi, -\xi)$

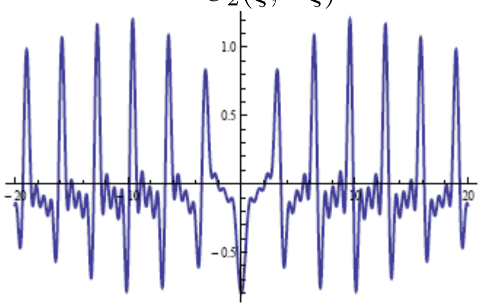
3 particles



ξa

$C_2(\xi, -\xi)$

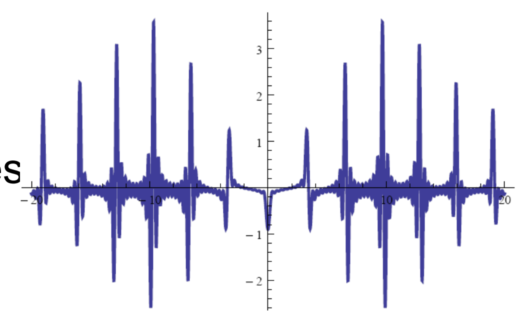
5 particles



ξa

$C_2(\xi, -\xi)$

10 particles



ξa

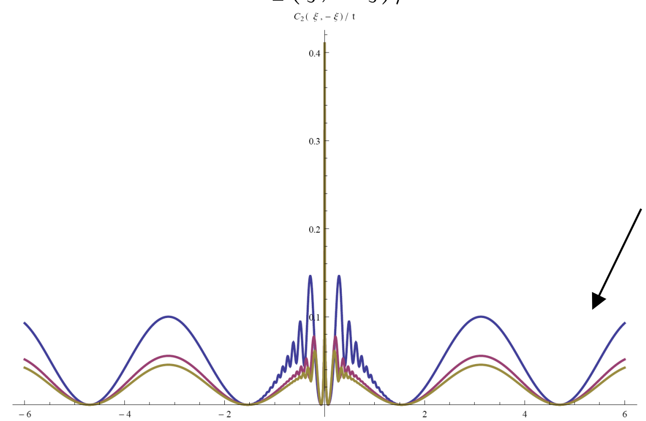
Fermionic dip as $\xi \to 0$

Structure emerges at $\xi a = \sigma$

Attractive bosons

2 particles

$C_2(\xi, -\xi)/t$



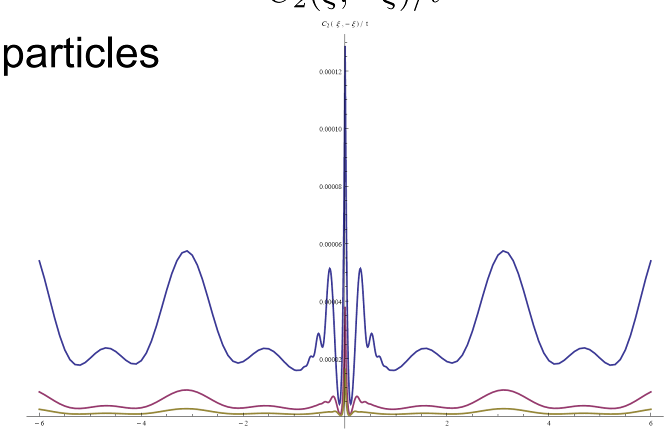
ξa

Blue - short times
Violet - longer
Magenta - longest

central peaks increase with time - weight in the bound states increases

$C_2(\xi, -\xi)/t$

3 particles



ξa

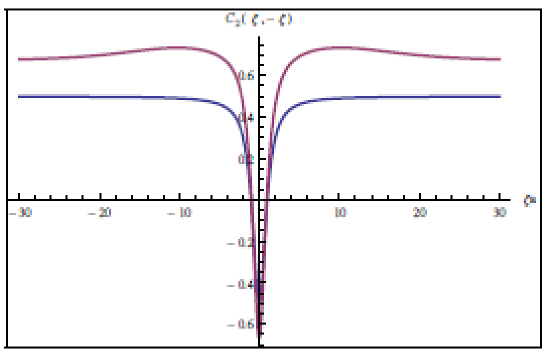
Blue - short times
Violet - longer
Magenta - longest

peaks diffuse – momenta redistribute

Evolution of a bosonic system: noise correlation

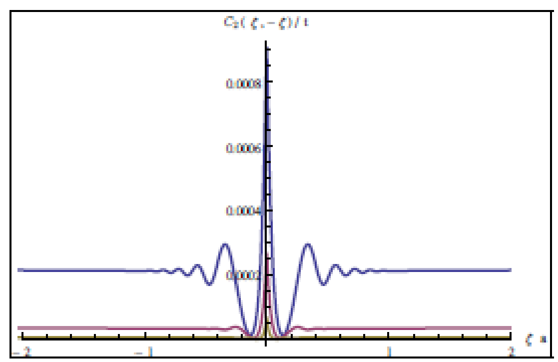
Noise correlations – starting from a condensate:

Repulsive bosons



Two (blue) and three bosons,

Attractive bosons



Three bosons, at times: $tc^2 = 20; 40; 60$

Emergence of an asymptotic Hamiltonian

Long time asymptotics - repulsive:

- **Bosons turn into fermions as time evolves (for any $c>0$)**

$$
\begin{aligned}
|\Phi_0,t\rangle &= \int_x\int_y\int_\lambda \theta(\vec{x})\Phi_0(\vec{x})\prod_{i<j}\frac{\lambda_i-\lambda_j-ic\,sgn(y_i-y_j)}{\lambda_i-\lambda_j-ic}\prod_j e^{-i\Sigma_j\lambda_j^2 t-\lambda_j(y_j-x_j)}\prod_j b^\dagger(y_j)|0\rangle \\
&= \int_x\int_y\int_\lambda \theta(\vec{x})\Phi_0(\vec{x})\prod_{i<j}\frac{\lambda_i-\lambda_j-ic\sqrt{t}\,sgn(y_i-y_j)}{\lambda_i-\lambda_j-ic\sqrt{t}}e^{-i\Sigma_j\lambda_j^2-\lambda_j(y_j-x_j)/\sqrt{t}}\prod_j b^\dagger(y_j)|0\rangle \\
&\to \int_x\int_y\int_\lambda \theta(\vec{x})\Phi_0(\vec{x})e^{-i\Sigma_j\lambda_j^2-\lambda_j(y_j-x_j)/\sqrt{t}}\prod_{i<j}sgn(y_i-y_j)\prod_j b^\dagger(y_j)|0\rangle \\
&= e^{-iH_0^f t}\int_{x,k}\mathcal{A}_x\,\theta(\vec{x})\Phi_0(\vec{x})\prod_j c^\dagger(x_j)|0\rangle.
\end{aligned}
$$

$\mathcal{A}_x$ *antisymmetrizer*

where

$H_0^f = -\int_x c^\dagger(x)\partial^2 c(c)$

- In the long time limit repulsive bosons for any $c>0$ propagate under the influence of* Tonks – Girardeau *Hamiltonian (hard core bosons=free fermions)

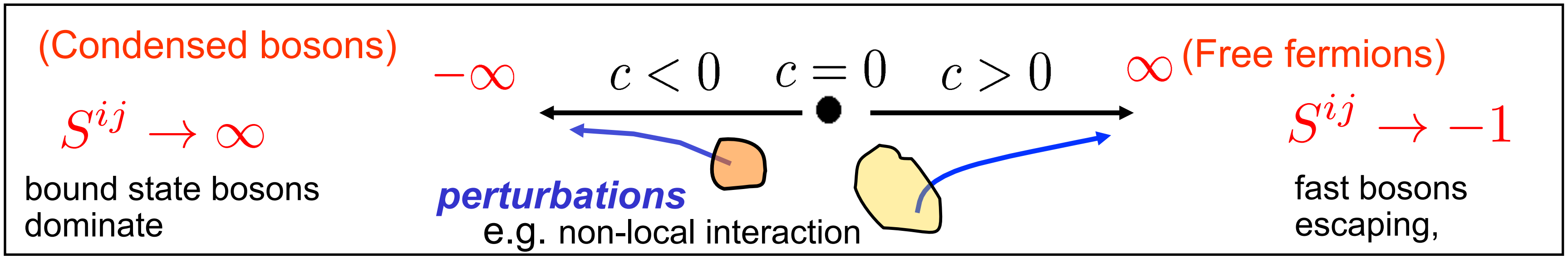
- The state equilibrates, does not thermalize

- Argument valid for any initial state Φ_0

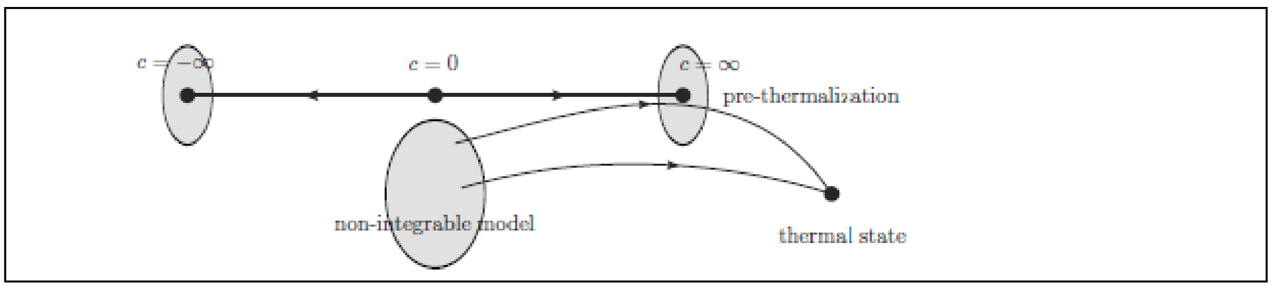
- **Scaling argument fails for attractive bosons** (instead, they form bound states)

Evolution of a bosonic system

- **Conclusion: coupling constant "*effectively evolves*" with time** $t \sim \ln(D_0/D)$

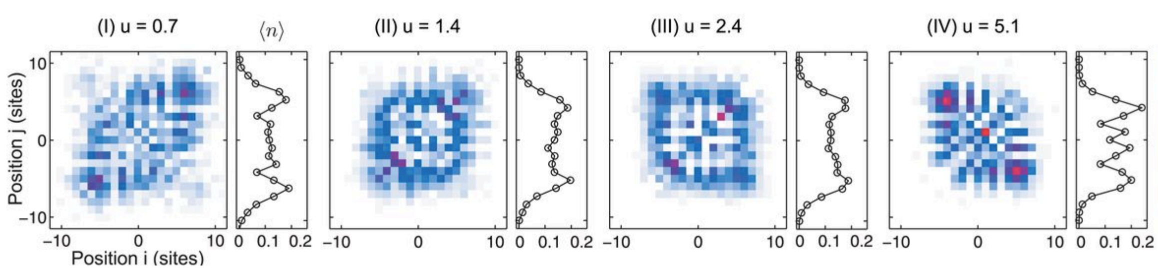


- **What is beyond $c = \infty$? Thermalization**

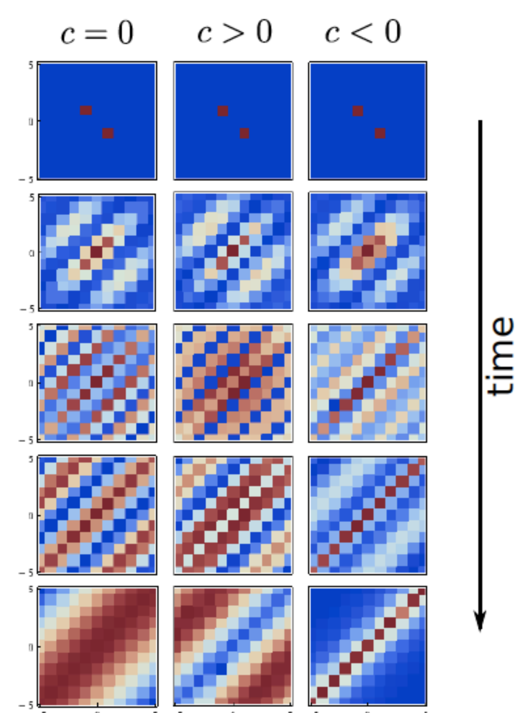


- **Fermionization also occurs on the lattice: *Bose Hubbard model* (not integrable)**

- **Observed experimentally recently** (*Greiner et al. '15*)



(Iyer, NA '13)



Evolution of the system in the thermodynamic regime

Thermodynamic regime - $N, L \to \infty, \;\; n = N/L \; fixed, \;\; t \ll L/v_{typ}$

To carry out time evolution need expand initial state $|\Phi^0\rangle$ in eigenstates $|k_1, \cdots, k_N\rangle$

Finite size Yudson representation: *(Goldstein, NA '13)*

Claim: Any initial state -

$$|\Phi_0\rangle = \int_{-L/2}^{L/2} d^N x \; \Phi_0\left(x_1, \cdots, x_N\right) b^\dagger\left(x_N\right), \cdots, b^\dagger\left(x_1\right)|0\rangle$$

Can be expressed in terms of eigenstates (a generalization of Yudson '85) $c > o$

$$|\Phi_0\rangle = \sum_{n_1=-\infty}^{\infty} \ldots \sum_{n_N=-\infty}^{\infty} \mathcal{N}\left(\{k_n\}\right)^{-1} |k_{n_1} \ldots . k_{n_N}\rangle \left(k_{n_1} \ldots k_{n_N}\right| |\Phi_0\rangle \,.$$

Overcomes the difficulty of calculating overlaps

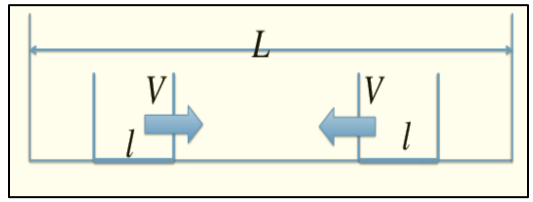
Time evolution- flow chart

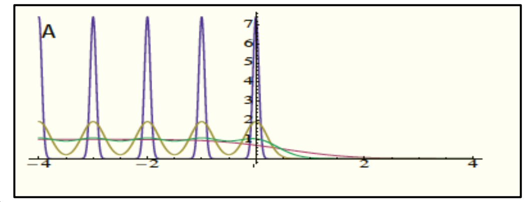
Initial state translational invariant

example: **Mott insulator**

Initial state not translational invariant

example: **Newton's cradle**, **Domain wall**



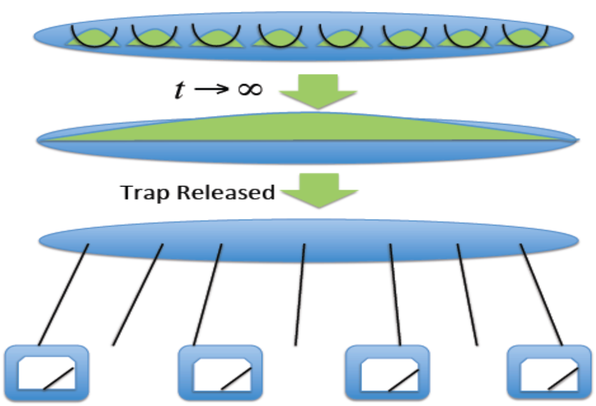


Equilibrates to local GGE (c >0)

Equilibrates but not to local GGE (c < 0)

Goldstein, NA '13

Universal correlations (if low YY entropy)



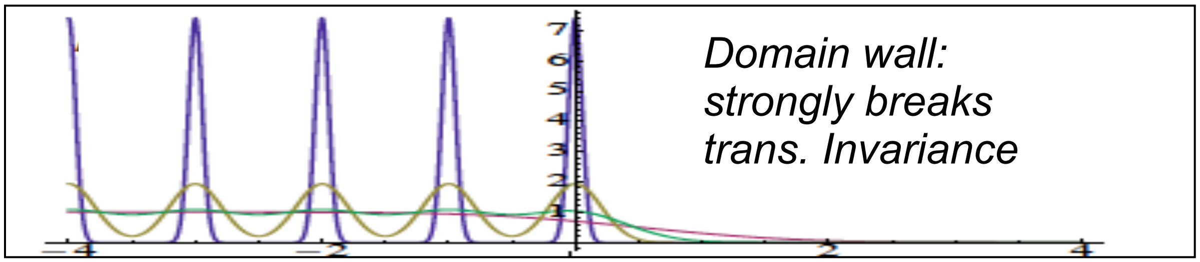
System does not equilibrate:
currents, *local* entropy production

Evolution to NESS: Domain wall

***Example**: time evolution from a non-trans. invariant initial state (no equilibration)*

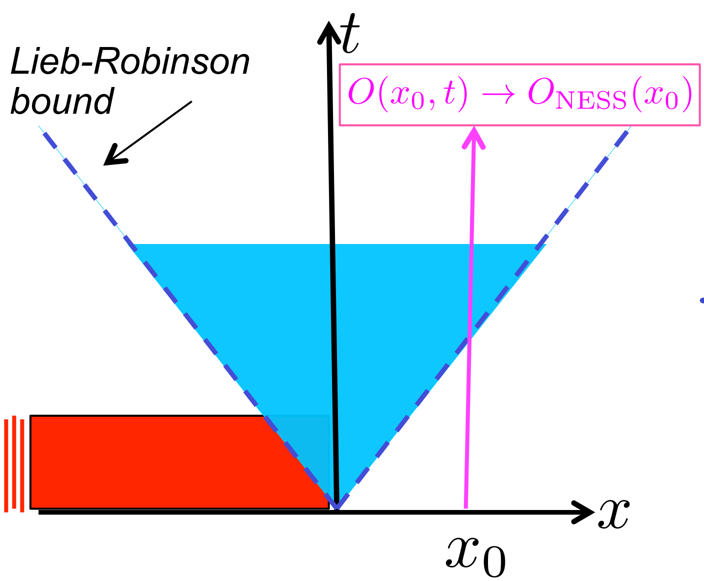
$$|\Psi(t=0)\rangle = \prod_{j=0}^{\infty}\int_{-\infty}^{\infty}\varphi(x+jl)\,b^{\dagger}(x)\,|0\rangle$$

with: $\varphi(x) = \dfrac{e^{-x^2/\sigma}}{(\pi\sigma/2)^{1/4}}$



- System evolves to NESS $\rho(x,t) \to$

(G Goldstein, NA '13)



$$\frac{1}{l}\left(1+\sum_{s=1}^{\infty}e^{-\pi^2(8t^2/\sigma+\sigma/2)s^2/l^2}\cos(2\pi s x/l)\right) \qquad x \ll -\sqrt{8t^2/\sigma}$$

Outside the cone

$$\frac{1}{l}\left(\frac{1}{2}-\frac{4\pi}{cl}\right) \qquad \sigma \ll |x| \ll t$$

Nonequilibrium steady state (NESS)
Independent of x, t; interaction effects

$$\frac{1}{l}\left(\frac{1}{2}Erfc\left(\frac{x}{\sqrt{A}}\right)+\right.$$
$$+\frac{16}{\pi cl}e^{-x^2/A}\left(\frac{1}{2}\sqrt{\pi}\frac{x}{\sqrt{A}}Erfc\left(\frac{x}{\sqrt{A}}\right)-\frac{1}{2}e^{-x^2/A}\right)$$
$$\left.+\frac{16}{\pi cl}\frac{\pi}{2}\left(1-\frac{1}{2}Erfc\left(\frac{x}{\sqrt{A}}\right)\right)Erfc\left(\frac{x}{\sqrt{A}}\right)\right) \qquad x \sim t$$

$A \equiv 8t^2/\sigma + \sigma/2$

Crossover regime

$$0 \qquad x \gg \sqrt{8t^2/\sigma}$$

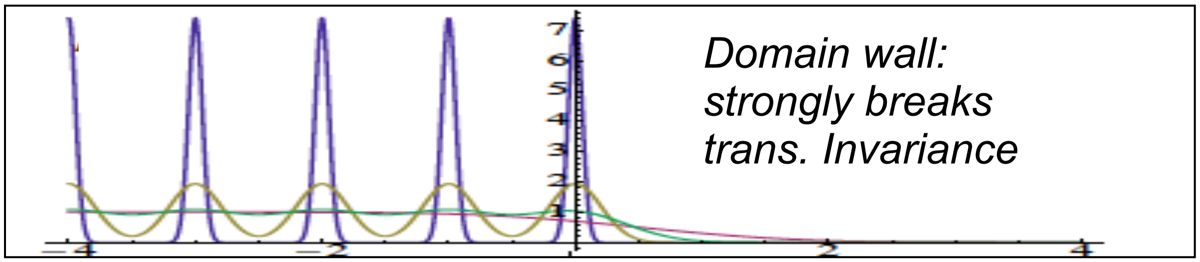
Outside the cone

Evolution to NESS: Domain wall (no thermalization)

***Example**: time evolution from a non-trans. invariant initial state (no equilibration)*

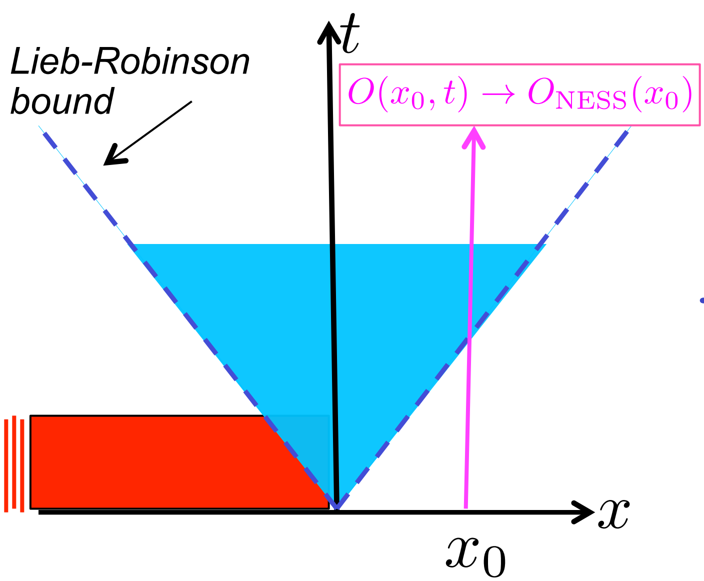
$$|\Psi(t=0)\rangle = \prod_{j=0}^{\infty} \int_{-\infty}^{\infty} \varphi(x+jl)\, b^{\dagger}(x)\, |0\rangle$$

with: $\varphi(x) = \dfrac{e^{-x^2/\sigma}}{(\pi\sigma/2)^{1/4}}$



- System evolves to NESS $\rho(x,t) \rightarrow$

(G Goldstein, NA '13)



$$\frac{1}{l}\left(1+\sum_{s=1}^{\infty} e^{-\pi^2(8t^2/\sigma+\sigma/2)s^2/l^2}\cos(2\pi s x/l)\right) \qquad x \ll -\sqrt{8t^2/\sigma}$$

Outside the cone

$$\frac{1}{l}\left(\frac{1}{2}-\frac{4\pi}{cl}\right) \qquad \sigma \ll |x| \ll t$$

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$$+\frac{16}{\pi cl}e^{-x^2/A}\left(\frac{1}{2}\sqrt{\pi}\frac{x}{\sqrt{A}}Erfc\left(\frac{x}{\sqrt{A}}\right)-\frac{1}{2}e^{-x^2/A}\right)$$
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$A \equiv 8t^2/\sigma + \sigma/2$

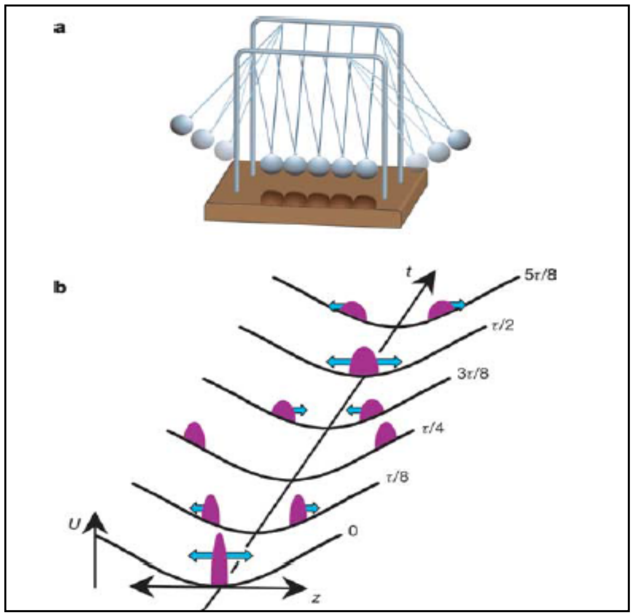
Crossover regime

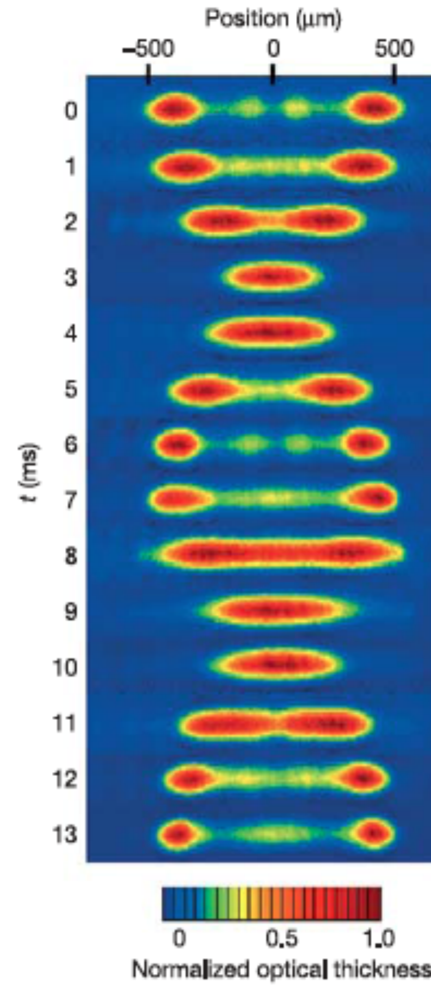
$$0 \qquad x \gg \sqrt{8t^2/\sigma}$$

Outside the cone

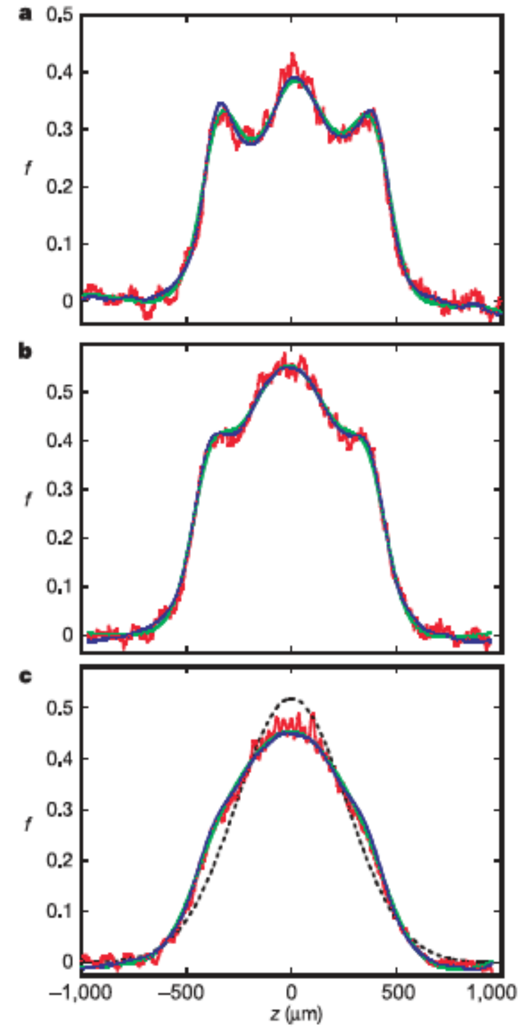
Newton's Cradle - simplified

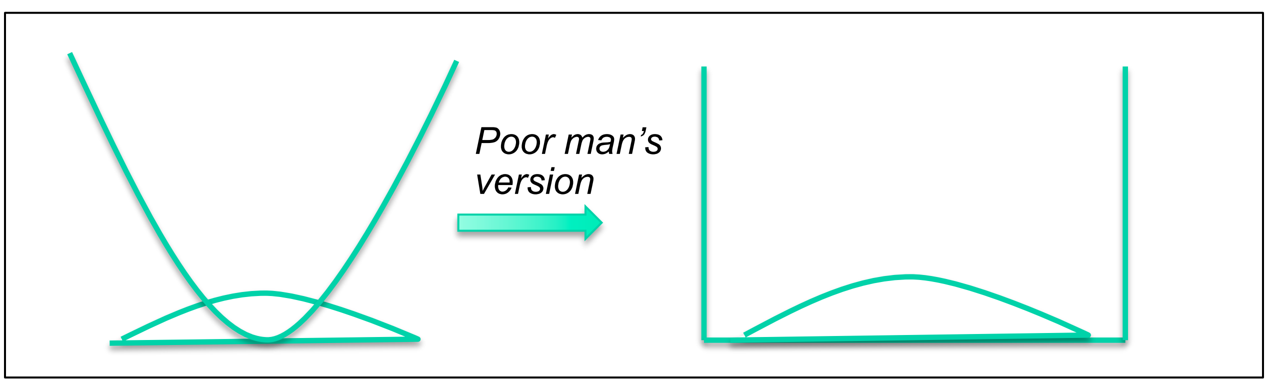
Kinoshita, T. Wenger, D. S. Weiss, Nature (2006)





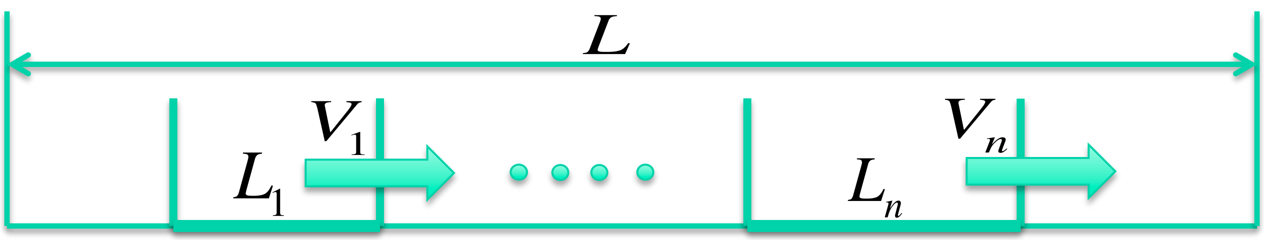
averaged momentum distributions





Newton's Cradle

Initial state: **several moving boxes within a larger container**
– *poor man's version of Weiss' experiment*



System does not equilibrate, but long time average is a diagonal ensemble

$$\langle \Theta \rangle_T \equiv \frac{1}{T}\int_0^T dt \langle \Psi | e^{iHt} \Theta e^{-iHt} | \Psi \rangle = \frac{1}{T}\sum_\lambda \sum_\kappa \frac{e^{i(E_\lambda - E_\kappa)T} - 1}{i\left(E_\lambda - E_\kappa\right)} \langle \Psi \mid \lambda \rangle \langle \lambda | \Theta | \kappa \rangle \langle \kappa \mid \Psi \rangle$$
$$\cong \sum_\lambda \langle \Psi \mid \lambda \rangle \langle \lambda | \Theta | \lambda \rangle \langle \lambda \mid \Psi \rangle$$

GGE for the average - *match* conservation laws:

$$L \int dk \rho_p^f(k) k^{2n} = \sum L_i \int \rho_p^i(k) \left(k + \frac{1}{2}V_i\right)^{2n}$$

LL in a box *(Gaudin)*

$$\psi\left(|k_1|,..|k_N|\right) = \sum_{\{\varepsilon\}} C\{\varepsilon\} \bar{\psi}\left(\varepsilon_1 |k_1|,..\varepsilon_N |k_N|\right)$$

All even charges are conserved $\{I_{2n}\}$

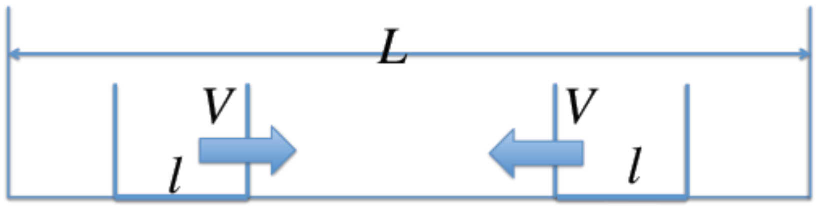
Solution for final quasi-particle density

$$\rho_p^f(k) = \sum \frac{L_i}{2L}\left(\rho_p^i\left(k + \frac{1}{2}V_i\right) + \rho_p^i\left(k - \frac{1}{2}V_i\right)\right)$$

Newton's Cradle in a box

Initial state:

Two boxes of length l each containing N bosons in a given initial state ρ^i moving towards each other at speed V



1. $$\rho^i_{gs}(k) = \theta(-k_F, k_F)\frac{1}{2\pi}\left(1+\frac{2k_F}{\pi c}\right)+o(\frac{k_F}{c})....$$

2. $$\rho^i_{\mathrm{BEC}}(x) = \frac{\tau\frac{d}{d\tau}a(x,\tau)}{1+a(x,\tau)}$$ $x=\frac{k}{c},\ \tau=\frac{n}{c}$ *Caux et al '12*

$$a(x,\tau) = \frac{2\pi\tau}{x\sinh(2\pi x)}J_{1-2ix}(4\sqrt{\tau})\,J_{1+2ix}(4\sqrt{\tau})$$

- The velocity distribution (measurable): $$P(v,t) = \int dx e^{-i\frac{v}{2}x}\left\langle b^\dagger(x)\,b(0)\right\rangle_t$$

- *The field-field correlation given in terms of the occupation probability* $f(k) = \frac{\rho_p(k)}{\rho_t(k)}$

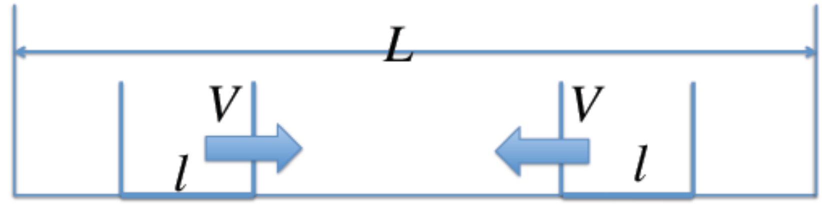
$$\left\langle b^\dagger(x)\,b(0)\right\rangle_{t\to\infty} = \int\frac{dk}{2\pi}f(k)\,e^{-ikx}\omega(k)\exp\left(-x\int du\,f(k)\,p_u(k)\right)$$ *Korepin, Izergin '87*

with:

$$2\pi p_u(k) = -\frac{k-u+ic}{u-k+ic}\exp\left(-\int ds\,f(s)\,K(u,s)\,p_s(k)\right)-1 \quad \omega(k)=\exp\left(-\frac{1}{2\pi}\int dqK(k,q)\,f(q)\right) \quad K(k,q)=\frac{2c}{(k-q)^2+c^2}$$

Newton's Cradle in a box

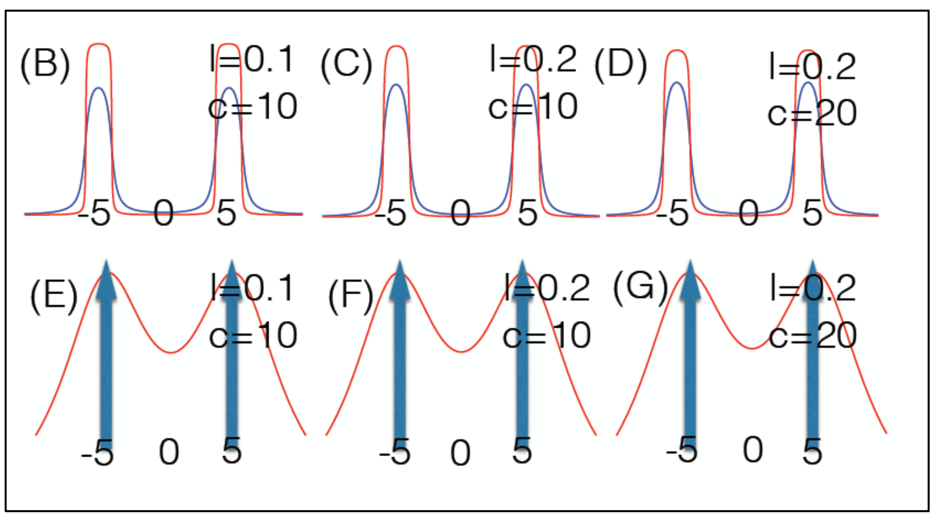
The velocity distribution:



$$P\left(v,t\right)=\int dx e^{-i\frac{v}{2}x}\left\langle b^{\dagger}\left(x\right)b\left(0\right)\right\rangle_{t}$$

ρ^{i} - ground state $\longrightarrow$ $P\left(v\right)\sim A_{L}\frac{\exp\left(-\frac{F_{L}}{\pi c}\right)}{2\pi}\sum_{i,j=\pm}\left(-1\right)^{j}\arctan A_{i,j}\left(v\right)$

ρ^{i} - BEC $\longrightarrow$ $P\left(v\right)\sim nB_{L}\frac{\exp\left(-\frac{G_{L}}{\pi c}\right)}{\pi}\left(\frac{H_{L}}{H_{L}^{2}+\frac{1}{4}\left(v-VK_{L}\right)}+\frac{H_{L}}{H_{L}^{2}+\frac{1}{4}\left(v+VK_{L}\right)}\right)$



(G Goldstein, NA '15)

$A_{\pm\pm}(v)=C_{L}\left(\left(1-F_{L}\frac{\exp\left(-2F_{L}/\pi c\right)}{\pi c}\right)\left(\pm\frac{V}{2}\pm k_{F}\right)+\frac{v}{2}\right)$ $F_{L}=4k_{F}A_{L}$ $K_{L}=\left(1-G_{L}\frac{\exp\left(-2G_{L}/\pi c\right)}{\pi c}\right)$ $C_{L}=\frac{2\pi}{4k_{F}A_{L}\left(1+\exp\left(-\frac{2F_{L}}{\pi c}\right)\right)}$

$A_{L}=\frac{l}{L}\left(1+\frac{2k_{F}}{\pi c}\left(1-\frac{2l}{L}\right)\right)$ $B_{L}=\frac{l}{L}\frac{1}{\frac{1}{2\pi}+\frac{2N}{\pi cL}}$ $G_{L}=2nB_{L}$ $H_{L}=\frac{G_{L}}{2\pi}\left(1+\exp\left(-\frac{2G_{L}}{\pi c}\right)\right)+2n\left(1-G_{L}\frac{\exp\left(-2G_{L}/\pi c\right)}{\pi c}\right)$

2. The Heisenberg Chain: Theory and Experiment

The XXZ Hamiltonian

$$H = J \sum_j \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta(\sigma_j^z \sigma_{j+1}^z - 1)$$

The phase diagram

$\Delta = -1$ $\quad LL \quad$ $\Delta = 1$

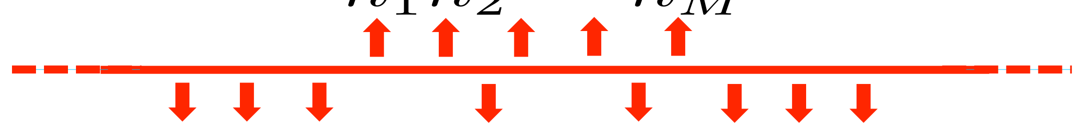
gapped, F — *critical regime* — *gapped, AF*

Quench from initial states: *Recall GGE fails*

$$|\Psi_0\rangle = \theta(n_1 > n_2 > \ldots n_N) \prod_j \sigma_{n_j}^+ |0\rangle$$

$n_1 n_2 \cdots n_M$

e. g.

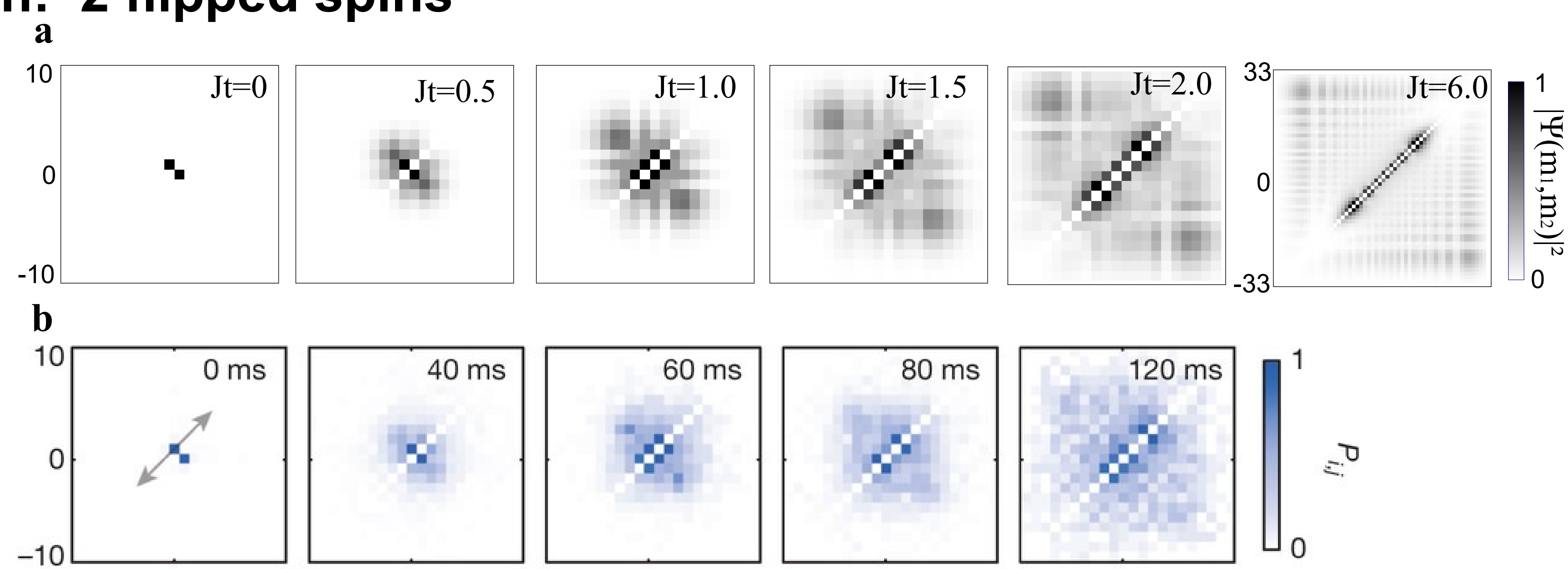


Time evolution: 2 flipped spins

Theory:

Experiment:

Munich group '13



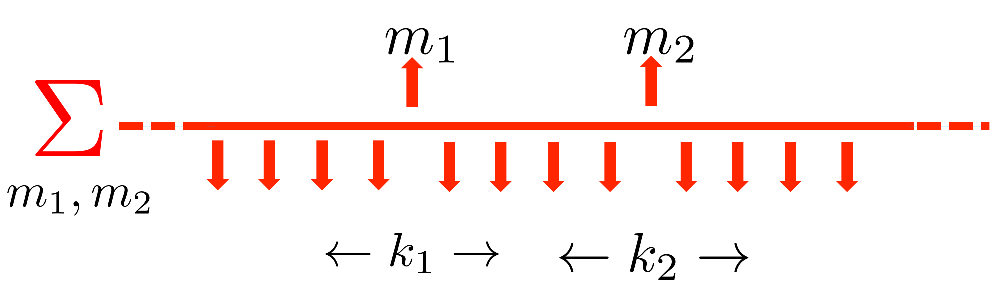
Eigenstates of the Heisenberg Chain

Eigenstates of the XXZ (M flipped spins)

$$|k\rangle = \sum_{\{m_j\}} \mathcal{S} \prod_{i<j} [\theta(m_i - m_j) + s(k_i, k_j)\theta(m_j - m_i)] \prod_j e^{ik_j m_j} \sigma^+_{m_j} |0\rangle$$

$$s(k_i, k_j) = e^{i\phi(k_i, k_j)} = -\frac{1 + e^{ik_i + ik_j} - 2\Delta e^{ik_i}}{1 + e^{ik_i + ik_j} - 2\Delta e^{ik_j}}$$

$$E = 4J \sum_{j=1}^{M} (\Delta - \cos k_j)$$



Time evolution of the XXZ magnet

i. Critical region $-1 < \Delta < 0$ $\quad \Delta = -\cos\mu \quad (0 < \mu < \frac{\pi}{2})$

Reparametrize: $\Delta \to \mu, \quad k \to \alpha$

$$e^{ik} \to \frac{\sinh\frac{i\mu-\alpha}{2}}{\sinh\frac{i\mu+\alpha}{2}} \longrightarrow$$

$$s(k_1,k_2) \to \frac{\sinh(\frac{\alpha_1-\alpha_2}{2} - i\mu)}{\sinh(\frac{\alpha_1-\alpha_2}{2} + i\mu)}$$

$$E(k) \to E(\alpha) = \frac{4J\sin^2\mu}{\cosh\alpha - \cos\mu}$$

The contour expression of the initial state:

$$|\Psi_0\rangle = \theta(n_1 > n_2 > ...n_N)\prod_j \sigma^+_{n_j}|0\rangle$$

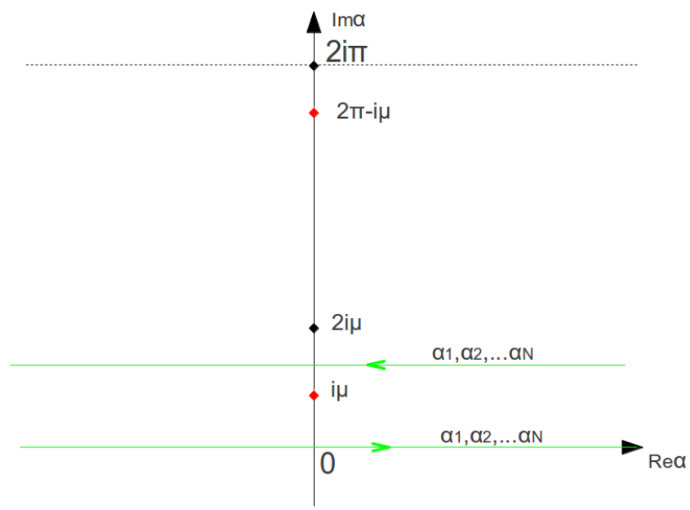
Expanded in terms of eigenstates

$$|\Psi_0\rangle = \sum_{\{m_j\}}\int_{\gamma_j}\prod_j\left[\frac{d\alpha_j}{2\pi}\frac{\sin\mu}{2\sinh\frac{\alpha_j+i\mu}{2}\sinh\frac{\alpha_j-i\mu}{2}}\right]\prod_j\left[\frac{\sinh(\frac{i\mu-\alpha_j}{2})}{\sinh(\frac{i\mu+\alpha_j}{2})}\right]^{m_j-n_j}$$

$$\times\prod_{i<j}\left[\theta(m_i-m_j) + \frac{\sinh(\frac{\alpha_i-\alpha_j}{2} - i\mu)}{\sinh(\frac{\alpha_i-\alpha_j}{2} + i\mu)}\theta(m_j-m_i)\right]\prod_j \sigma^+_{m_j}|0\rangle$$

Evolution of the XXZ magnet

The contour:



The time evolved state:

$$|\Psi(t)\rangle = \sum_{\{m_j\}} \int_{\gamma_j} \prod_j [\frac{d\alpha_j}{2\pi} \frac{\sinh\lambda}{2\sin\frac{\alpha_j+i\lambda}{2}\sin\frac{\alpha_j-i\lambda}{2}}] \prod_{i<j} [\theta(m_i-m_j) + \frac{\sin(\frac{\alpha_i-\alpha_j}{2}-i\lambda)}{\sin(\frac{\alpha_i-\alpha_j}{2}+i\lambda)}\theta(m_j-m_i)]$$

$$\times \prod_j [\frac{\sin(\frac{i\lambda-\alpha_j}{2})}{\sin(\frac{i\lambda+\alpha_j}{2})}]^{m_j-n_j} e^{-iE(\alpha_j)t} \sigma^+_{m_j} |0\rangle$$

Evolution of the XXZ magnet

ii. $\Delta < -1$ *Ferromagnetic regime*

$$\Delta = -\cosh\lambda \to \lambda > 0$$

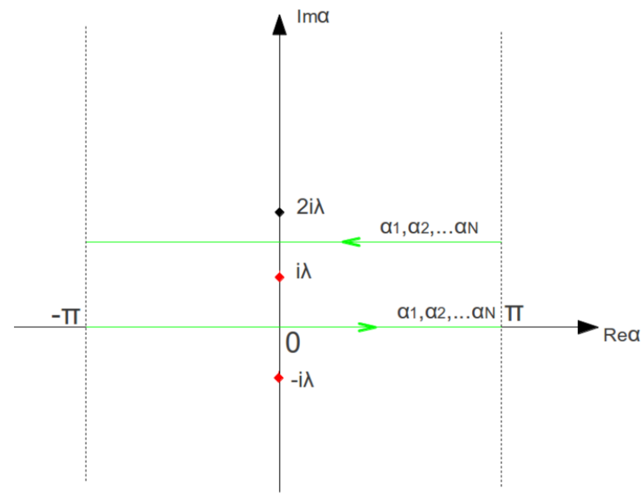
Reparametrize:

$$e^{ik} \to \frac{\sin\frac{i\lambda-\alpha}{2}}{\sin\frac{i\lambda+\alpha}{2}}$$

$$s(k_1,k_2) \to \frac{\sin(\frac{\alpha_1-\alpha_2}{2}-i\lambda)}{\sin(\frac{\alpha_1-\alpha_2}{2}+i\lambda)}$$

$$E(k) \to E(\alpha) = -\frac{4J\sinh^2\lambda}{\cos\alpha - \cosh\lambda}$$

The contour:



The time evolved state

$$|\Psi(t)\rangle = \sum_{\{m_j\}} \int_{\gamma_j} \prod_j [\frac{d\alpha_j}{2\pi} \frac{\sinh\lambda}{2\sin\frac{\alpha_j+i\lambda}{2}\sin\frac{\alpha_j-i\lambda}{2}}] \prod_{i<j} [\theta(m_i-m_j) + \frac{\sin(\frac{\alpha_i-\alpha_j}{2}-i\lambda)}{\sin(\frac{\alpha_i-\alpha_j}{2}+i\lambda)}\theta(m_j-m_i)]$$

$$\times \prod_j [\frac{\sin(\frac{i\lambda-\alpha_j}{2})}{\sin(\frac{i\lambda+\alpha_j}{2})}]^{m_j-n_j} e^{-iE(\alpha_j)t}\sigma^+_{m_j}|0\rangle$$

Evolution of the XXZ magnet

Some results *- local magnetization and bound states*

- Spin currents

Start from

$$|\Psi_0\rangle = \sigma^-_{-1}\sigma^-_0\sigma^-_{+1}|\Uparrow\rangle$$

Calculate:

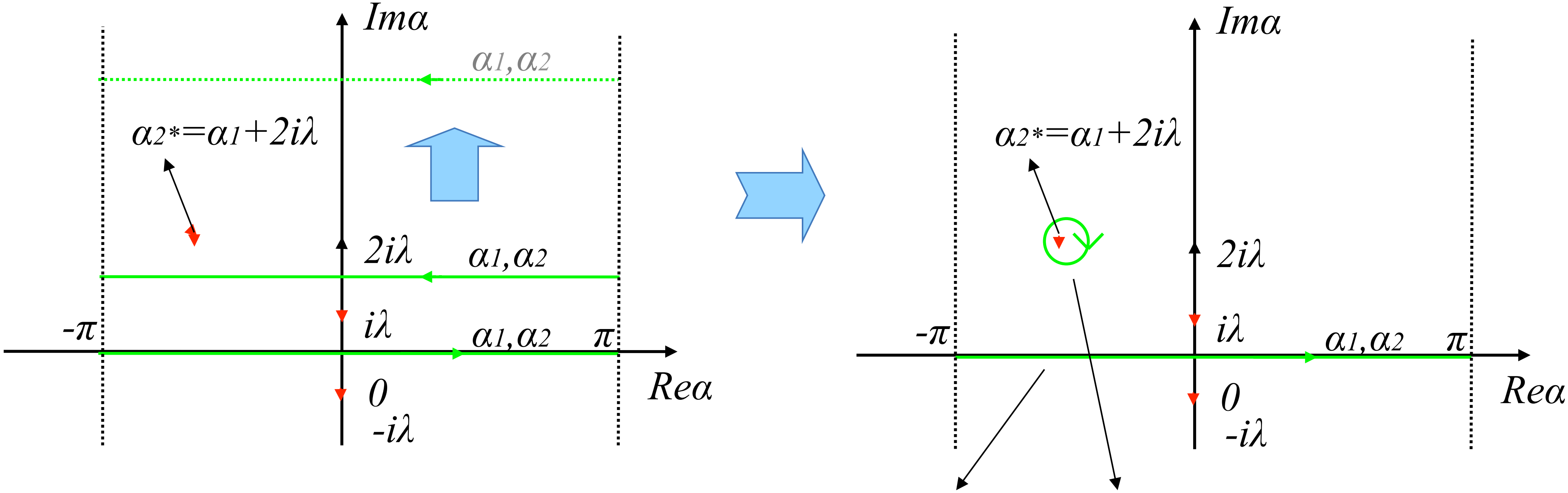
$$M(n,t) = \langle\Psi(t)|\sigma^z_n|\Psi(t)\rangle$$

$$I(n,t) = i\langle\Psi(t)|(\sigma^+_n\sigma^-_{n+1} - \sigma^-_n\sigma^+_{n+1})|\Psi(t)\rangle$$

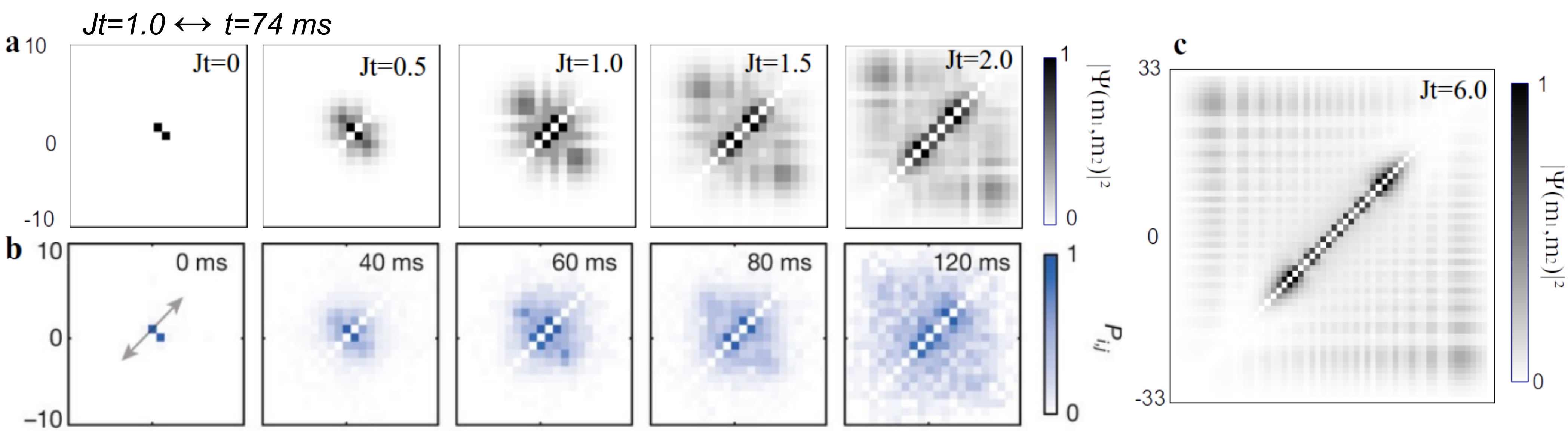
For different values of anisotropy Δ

- as the anisotropy increases the weight of the bound states increases

Contour Shift and Bound States



$$\Psi^{1,0}(m_1, m_2; t) = \Psi_{magn}(m_1, m_2; t) + \Psi_{bound}(m_1, m_2; t)$$

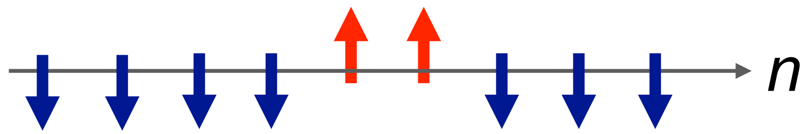


b. T. Fukuhara et al, Nature **502**, 76 (2013)

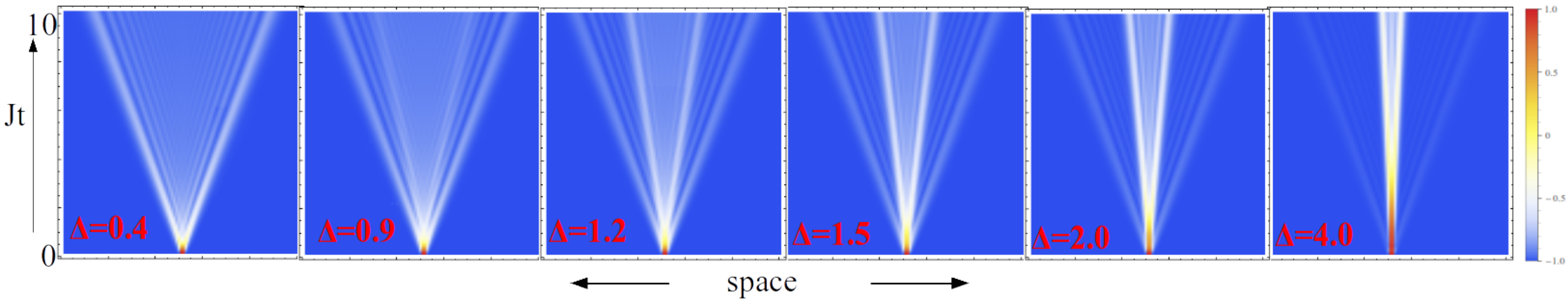
Observables

- **Local Magnetization** $M(n,t) \equiv \langle \Psi(t)|\sigma_n^z|\Psi(t)\rangle$

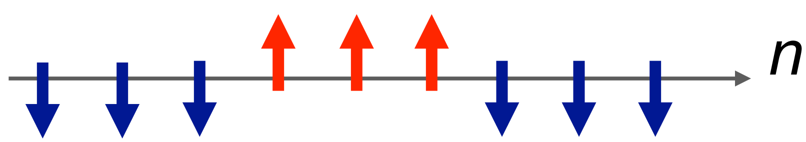
$$|\Psi_0\rangle = \sigma_1^+\sigma_0^+|\Downarrow\rangle =$$

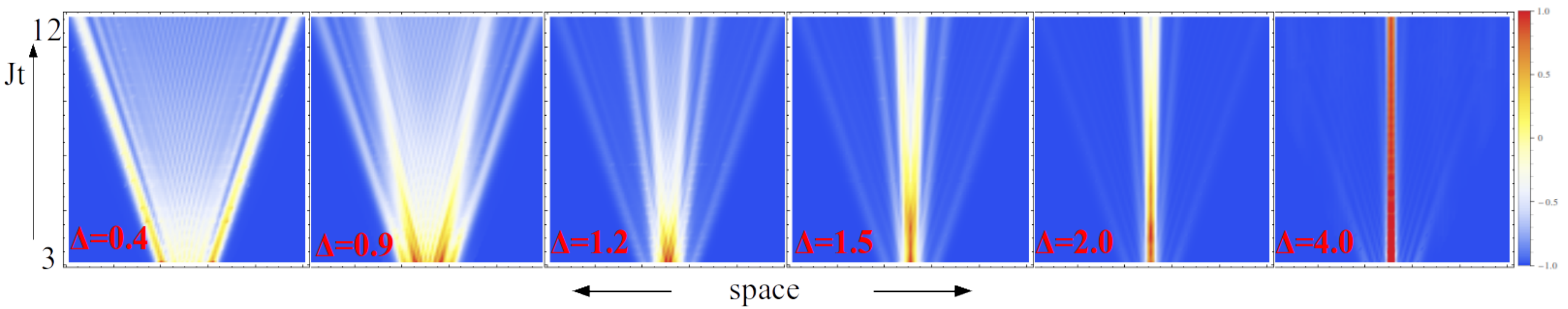


(cf. Ganahl et al. '12)



$$|\Psi_0\rangle = \sigma_1^+\sigma_0^+\sigma_{-1}^+|\Downarrow\rangle =$$



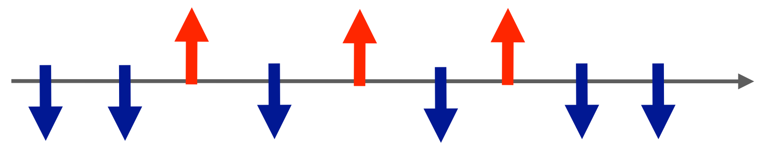


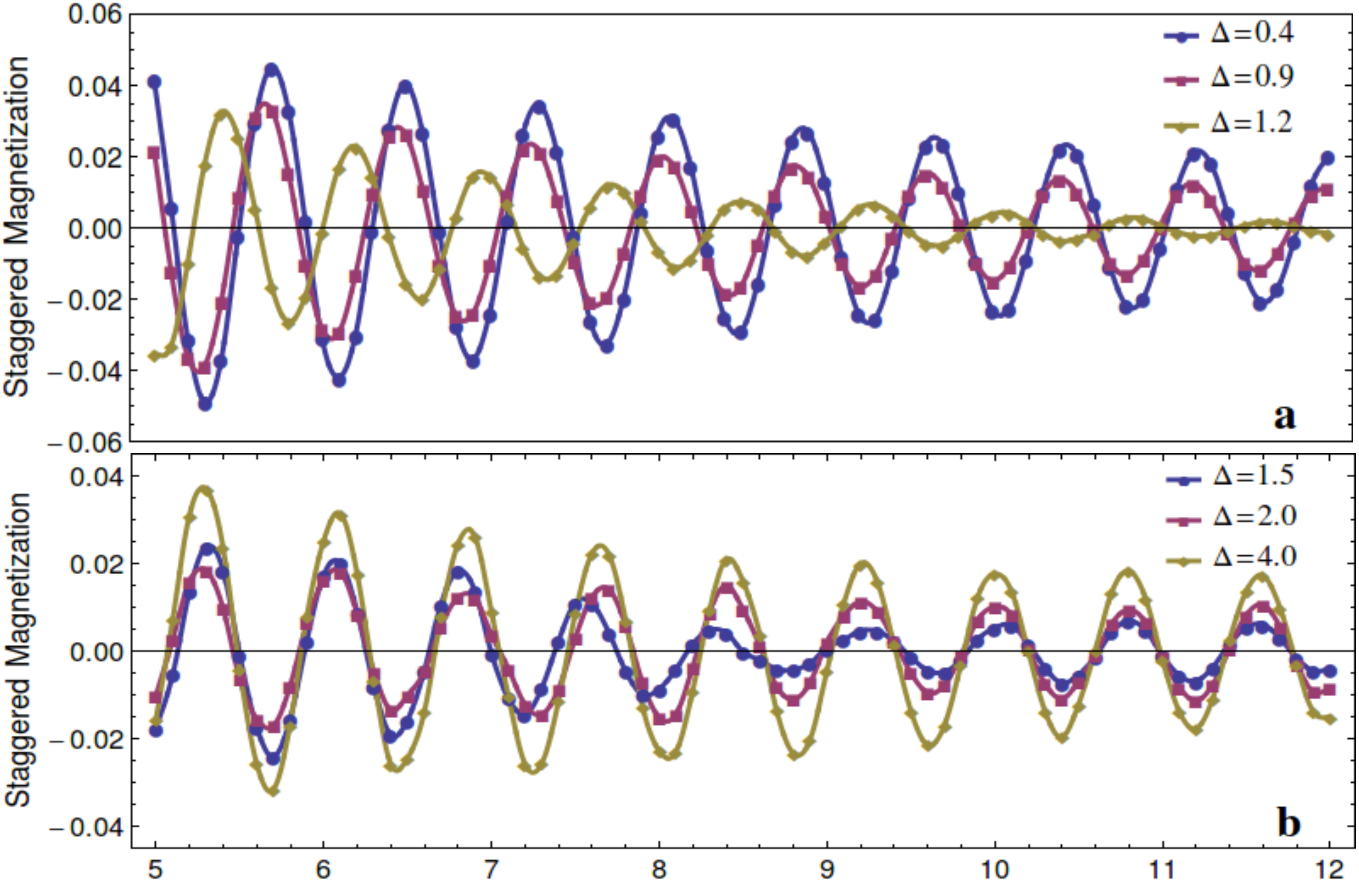
$$\frac{v_{bound}}{v_{magn}} = \frac{\sin\mu}{\sin(n\mu)}(|\Delta| = \cos\mu) \qquad \frac{v_{bound}}{v_{magn}} = \frac{\sinh\lambda}{\sinh(n\lambda)}(|\Delta| = \cosh\lambda)$$

- *<u>Staggerd Magnetization (Order Parameter)</u>* $M_s(t) = \frac{1}{N}\sum_n (-1)^n \langle \sigma_n^z \rangle (t)$

Quench across a QCP $\Delta = \infty \rightarrow |\Delta| < 1$

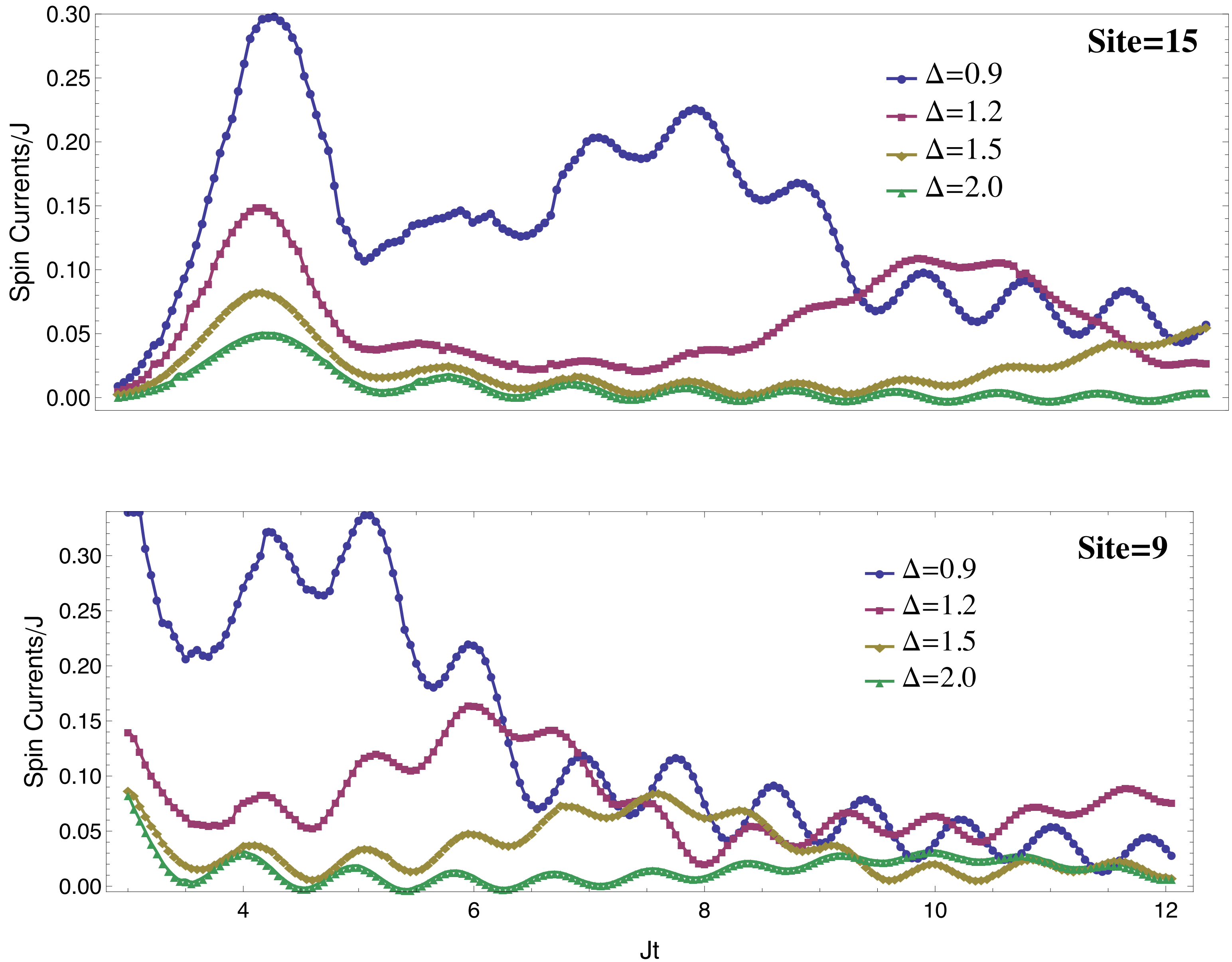
$|\Psi_0\rangle = \sigma \quad \sigma_0^+ \sigma \quad _2 | \Downarrow \rangle$





Evolution of the XXZ magnet

Spin currents - evolution



Outlook

Conclusions:

- **Evolution calculable for all coupling regimes – in thermodynamic regime (or non-therm)**
- **Evolution for all initial states** (asymptotic equilibrium or not**)**

To do list:

- **Quench dynamics in other integrable models:**

Anderson model **(*Adrian Culver*),** *Lieb-Liniger + impurity, Gaudin-Yang* **(*Huijie Guan*),**
Kane-Fisher, Luttinger- quantum dot systems, Luttinger-Kondo **(*Colin Rylands*),**
Kondo dynamics **(*Roshan Tourani*),** spin impurity at the edge of a TI (Parmesh Pasnoori)

- **Floquet dynamics in Lieb-liniger model (Vaibhav Dwivedi)**

- **nonequilibrium transport across quantum dots (Adrian Culver), nonequilibrium transport across multi-dot system dot (Caitlin Carpenter)**

- **Scattering cross sections of an electron off a Kondo system (Chris Munson)**

Big Questions: *(Being Boltzmann?)*

- **What drives thermalization of pure states? Canonical typicality, entanglement entropy (Lebowitz, Tasaki, Short…)**

- **General principles, variational? F-D theorem out-of-equilibrium? Heating? Entanglement?**

- **What is universal? RG Classification?**