On the topological quantum integrable systems

Yupeng Wang

Institute of Physics, CAS

Collaborators : J. Cao, W.L. Yang, K. Shi, S. Cui, X. Zhang,
Y.-Y. Li

2017, Mar. 27-29/ Beijing

Outline

I. Motivation
II. The inhomogeneous T-Q relation
III. The topological spin chain
IV. The XYZ model
V. Concluding remarks & perspective

I. Motivation

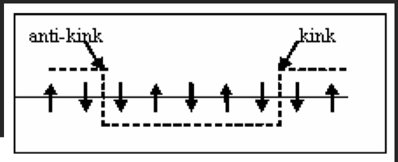
Why exactly solvable models?

Provide important benchmarks!

Hydrogen atom

2D Ising model → thermodynamic phase transition

1D Hubbard model → Mott insulator



Heisenberg chain → Spinon （fractional charge）

Recent applications：Cold atoms、Yang-Mills and AdS/CFT、Stochastic process、Topological states of matter、Quantum information

Topology becomes very important!

I. Motivation

Study on topological quantum integrable models initiated from Baxter's work on the XYZ model

$$H=\frac{1}{2}\sum_{n=1}^{N}\left(J_x\sigma_n^x\sigma_{n+1}^x+J_y\sigma_n^y\sigma_{n+1}^y+J_z\sigma_n^z\sigma_{n+1}^z\right)$$

Z_2 operators:

$$U^i=\sigma_1^i\sigma_2^i\cdots\sigma_N^i,\quad i=x,y,z.$$

$$(U^i)^2=\mathrm{id},\quad U^i\,U^j=(-1)^N U^j\,U^i,\ \text{for } i\neq j$$

The odd N case kept unsolved stubbornly for 40 years!

I. Motivation

Two classes of integrable models

Ordinary with U(1)

Periodic boundary
Parallel boundary
.....

Coordinate BA,
Baxter's T-Q,
Algebraic BA

Topologically nontrivial
Without U(1) symmetry

XYZ spin chain (odd N)
Anti-periodic boundary
Non-diagonal boundary fields
Cyclic representation……

Off-diagonal Bethe Ansatz

I. Motivation

Baxter's T-Q relation

$$\mathbf{t}(u) = a(u)\frac{\mathbf{Q}(u-\eta)}{\mathbf{Q}(u)} + d(u)\frac{\mathbf{Q}(u+\eta)}{\mathbf{Q}(u)}$$

$$[\mathbf{t}(u), \mathbf{Q}(v)] = [\mathbf{Q}(u), \mathbf{Q}(v)] = 0$$

$$\mathbf{t}(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle$$

$$\mathbf{Q}(u)|\Psi\rangle = Q(u)|\Psi\rangle$$

$$\Lambda(u) = a(u)\frac{Q(u-\eta)}{Q(u)} + d(u)\frac{Q(u+\eta)}{Q(u)}$$

$$Q(u) = \prod_{j=1}^{M} f(u-\lambda_j), \qquad f(0) = 0$$

Regularity

$$a(\lambda_j)Q(\lambda_j-\eta) + d(\lambda_j)Q(\lambda_j+\eta) = 0$$

I. Motivation

$$\mathbf{t}(u) = a(u)\frac{\mathbf{Q}(u-\eta)}{\mathbf{Q}(u)} + d(u)\frac{\mathbf{Q}(u+\eta)}{\mathbf{Q}(u)} + \frac{F(u)}{\mathbf{Q}(u)}$$

Nontrivial topology!

Why polynomial Q? (1) Simple BAEs; (2) Easy thermodynamic limit (integrals).

However, for topologically nontrivial models, there is no polynomial Q-solution!
It violates either asymptotic behavior or periodicity

Boundary becomes very important!
Edge states in QH, TI etc.

II. The inhomogeneous T-Q relation

Absence of reference state !

Functional analysis!

The eigenvalue of the transfer matrix is a degree N polynomial!

$$\Lambda(u) = c\prod_{j=1}^{N} f(u - z_j)$$

N+1 equations or values at N+1 points $\Lambda(\theta_j)$ determine it completely!

The operator product identities

Consider an R-matrix

$$R_{0,j}(u) = u + \eta P_{0,j} = u + \frac{1}{2}\eta(1 + \sigma_j \cdot \sigma_0)$$

Intrinsic properties:

$$\begin{aligned}
&\text{Initial condition}: \; R_{1,2}(0) = P_{1,2}, \\
&\text{Unitary relation}: \; R_{1,2}(u)R_{2,1}(-u) = -\varphi(u) \times \mathrm{id}, \\
&\qquad \varphi(u) = u^2 - 1, \\
&\text{Crossing relation}: \; R_{1,2}(u) = -\sigma_1^y R_{1,2}^{t_1}(-u-1)\sigma_1^y,
\end{aligned}$$

Transfer matrix:

$$\begin{aligned}
T_0(u) &= R_{0,N}(u-\theta_N)\cdots R_{0,1}(u-\theta_1), \\
t(u) &= tr_0 T_0(u),
\end{aligned}$$

$$[t(u),\ t(v)] = 0$$

$$\begin{aligned}
t(\theta_j) &= tr_0\{R_{0,N}(\theta_j-\theta_N)\cdots R_{0,j+1}(\theta_j-\theta_{j+1}) \\
&\quad \times P_{0,j}R_{0,j-1}(\theta_j-\theta_{j-1})\cdots R_{0,1}(\theta_j-\theta_1)\} \\
&= R_{j,j-1}(\theta_j-\theta_{j-1})\cdots R_{j,1}(\theta_j-\theta_1) \\
&\quad \times tr_0\{R_{0,N}(\theta_j-\theta_N)\cdots R_{0,j+1}(\theta_j-\theta_{j+1})P_{0,j}\} \\
&= R_{j,j-1}(\theta_j-\theta_{j-1})\cdots R_{j,1}(\theta_j-\theta_1) \\
&\quad \times R_{j,N}(\theta_j-\theta_N)\cdots R_{j,j+1}(\theta_j-\theta_{j+1}).
\end{aligned}$$

$$\begin{aligned}
t(\theta_j-1) &= tr_0\{R_{0,N}(\theta_j-\theta_N-1)\cdots R_{0,1}(\theta_j-\theta_1-1)\} \\
&= (-1)^N\, tr_0\{\sigma_0^y R_{0,N}^{t_0}(-\theta_j+\theta_N)\cdots R_{0,1}^{t_0}(-\theta_j+\theta_1)\sigma_0^y\} \\
&= (-1)^N\, tr_0\{R_{0,1}(-\theta_j+\theta_1)\cdots R_{0,N}(-\theta_j+\theta_N)\} \\
&= (-1)^N R_{j,j+1}(-\theta_j+\theta_{j+1})\cdots R_{j,N}(-\theta_j+\theta_N) \\
&\quad \times R_{j,1}(-\theta_j+\theta_1)\cdots R_{j,j-1}(-\theta_j+\theta_{j-1}).
\end{aligned}$$

$$\begin{aligned}
&t(\theta_j)t(\theta_j-1) = a(\theta_j)d(\theta_j-1), \quad j=1,\ldots,N, \\
&a(u) = \prod_{j=1}^{N}(u-\theta_j+1), \quad d(u) = \prod_{j=1}^{N}(u-\theta_j).
\end{aligned}$$

$\xrightarrow{\text{Homogeneous}}$

$$\frac{\partial^l}{\partial u^l}\{t(u)t(u-1)-a(u)d(u-1)\}|_{u=0,\{\theta_j=0\}} = 0, \quad l=0,\ldots,N-1.$$

II. The inhomogeneous T-Q relation

$$\mathrm{t}(\theta_j)\mathrm{t}(\theta_j-\eta)=a(\theta_j)d(\theta_j-\eta)\times id\sim\Delta_q(\theta_j),\quad j=1,\cdots,N$$

$$a(\theta_j-\eta)=d(\theta_j)=0$$

$$\mathrm{t}(u)|\Psi\rangle=\Lambda(u)|\Psi\rangle$$

$$\Lambda(\theta_j)\Lambda(\theta_j-\eta)=a(\theta_j)d(\theta_j-\eta)$$

$$\begin{aligned}\Lambda(u) &= a(u)\frac{Q(u-\eta)}{Q(u)}+d(u)\frac{Q(u+\eta)}{Q(u)}+c(u)\frac{a(u)d(u)}{Q(u)}\\ Q(u) &= \prod_{j=1}^{M} f(u-\lambda_j)\end{aligned}$$

C(u) is nonsingular and matches asymptotic behavior or periodicity

Why?

$$\Lambda(\theta_j)=a(\theta_j)\frac{Q(\theta_j-\eta)}{Q(\theta_j)}\quad\times\quad\Lambda(\theta_j-\eta)=d(\theta_j-\eta)\frac{Q(\theta_j)}{Q(\theta_j-\eta)}$$

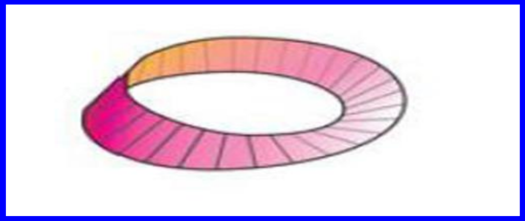
Regularity

$$a(\lambda_j)Q(\lambda_j-\eta)+d(\lambda_j)Q(\lambda_j+\eta)+c(\lambda_j)a(\lambda_j)d(\lambda_j)=0$$

III. The topological spin chain

Cao et. al, PRL 111, 137201(2013)

$$H = -\sum_{j=1}^{N}\left[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \cosh\eta\,\sigma_j^z\sigma_{j+1}^z\right]$$



$$\sigma_{N+1}^{\alpha} = \sigma_1^x\sigma_1^{\alpha}\sigma_1^x$$

Monodromy matrix

$$T_0(u) = \sigma_0^x R_{0,N}(u-\theta_N)\cdots R_{0,1}(u-\theta_1) = \begin{pmatrix} C(u) & D(u) \\ A(u) & B(u) \end{pmatrix}$$

Transfer matrix

$$t(u) = tr_0 T_0(u) = B(u) + C(u)$$

$$t(\theta_j)t(\theta_j-\eta) = -a(\theta_j)d(\theta_j-\eta), \quad j = 1,\cdots,N,$$

$$d(u) = a(u-\eta) = \prod_{j=1}^{N}\frac{\sinh(u-\theta_j)}{\sinh\eta}$$

III. The topological spin chain

Functional relation

$$\Lambda(\theta_j)\Lambda(\theta_j-\eta) = -a(\theta_j)d(\theta_j-\eta), \quad j=1,\cdots,N.$$

Periodicity

$$\Lambda(u+i\pi) = (-1)^{N-1}\Lambda(u)$$

Degree N-1 trigonometric polynomial

$$\Lambda(u) = a(u)e^{u}\frac{Q(u-\eta)}{Q(u)} - e^{-u-\eta}d(u)\frac{Q(u+\eta)}{Q(u)} - c(u)\frac{a(u)d(u)}{Q(u)}$$

$$Q(u) = \prod_{j=1}^{N}\sinh(u-\lambda_j),$$

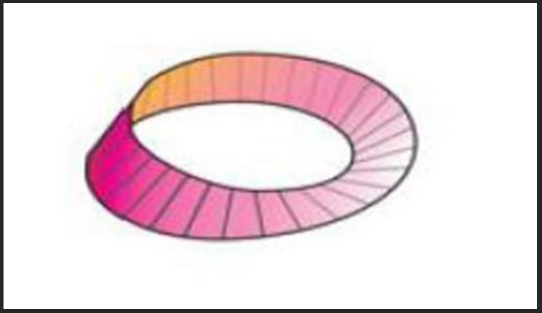
$$c(u) = \sinh^N\eta\left[e^{u-N\eta+\sum_{j=1}^{N}(\theta_j-\lambda_j)} - e^{-u-\eta-\sum_{j=1}^{N}(\theta_j-\lambda_j)}\right].$$

BAE

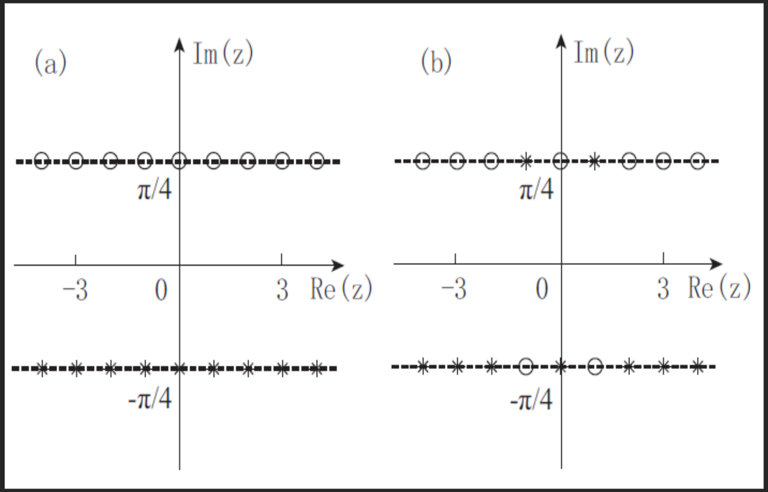
$$e^{\lambda_j}a(\lambda_j)Q(\lambda_j-\eta) - e^{-\lambda_j-\eta}d(\lambda_j)Q(\lambda_j+\eta) - c(\lambda_j)a(\lambda_j)d(\lambda_j) = 0,$$
$$j=1,\ldots,N.$$

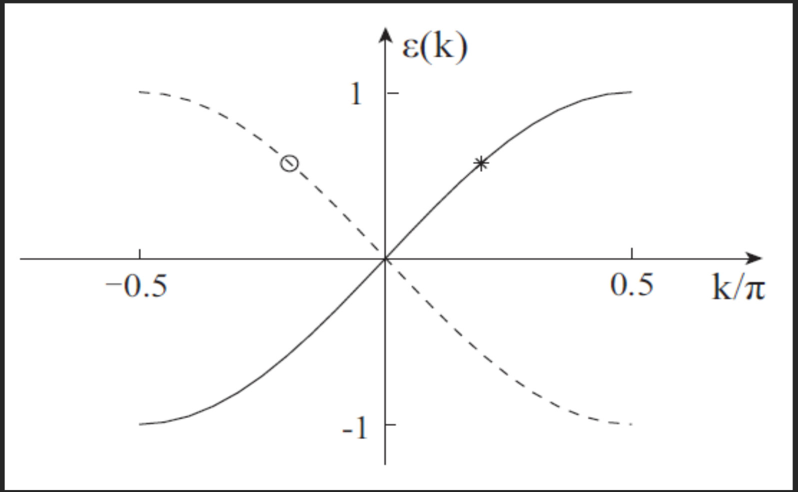
III. The topological fermion chain PRL 111,137201(2013)

$$H = -2\sum_{j=1}^{N-1}\left\{a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j\right\} - 2U^z\left[a_1^\dagger a_N^\dagger + a_N a_1\right]$$



Superconducting quantum dot embedded in a metallic ring





Gapless：topological particle-hole excitation
Gapped: bulk zero mode

III. The topological spin chain: Bethe states

A convenient basis

$$\langle\theta_{p_1},\cdots,\theta_{p_n}| = \langle 0|\prod_{j=1}^{n} C(\theta_{p_j}),$$

$$|\theta_{q_1},\cdots,\theta_{q_n}\rangle = \prod_{j=1}^{n} B(\theta_{q_j})|0\rangle,$$

$$D(u)|\theta_{p_1},\cdots,\theta_{p_n}\rangle = d(u)\prod_{j=1}^{n}\frac{\sinh(u-\theta_{p_j}+\eta)}{\sinh(u-\theta_{p_j})}|\theta_{p_1},\cdots,\theta_{p_n}\rangle,$$

$$\langle\theta_{p_1},\cdots,\theta_{p_n}|D(u) = d(u)\prod_{j=1}^{n}\frac{\sinh(u-\theta_{p_j}+\eta)}{\sinh(u-\theta_{p_j})}\langle\theta_{p_1},\cdots,\theta_{p_n}|.$$

$q_j, p_j \in (1,\cdots,N)$, $p_1 < p_2 < \cdots < p_n$ and $q_1 < q_2 < \cdots < q_n$.

$$\langle\theta_{p_1},\cdots,\theta_{p_n}|\theta_{q_1},\cdots,\theta_{q_m}\rangle = f_n(\theta_{p_1},\cdots,\theta_{p_n})\delta_{m,n}\prod_{j=1}^{n}\delta_{p_j,q_j},$$

$$f_n(\theta_{p_1},\cdots,\theta_{p_n}) = \prod_{j=1}^{n} a(\theta_{p_j})d_{p_j}(\theta_{p_j})\prod_{k\neq l}^{n}\frac{\sinh(\theta_{p_k}-\theta_{p_l}+\eta)}{\sinh(\theta_{p_k}-\theta_{p_l})}$$

Orthogonal and complete basis

III. The topological spin chain: Bethe states

Scalar product

$$F_n(\theta_1,\cdots,\theta_n)=\langle\theta_1,\cdots,\theta_n|\Psi\rangle$$

$$\Lambda(\theta_{n+1})F_n(\theta_1,\cdots,\theta_n)=\langle\theta_1,\cdots,\theta_n|t(\theta_{n+1})|\Psi\rangle=F_{n+1}(\theta_1,\cdots,\theta_{n+1})$$

$$\Downarrow$$

$$F_n(\theta_1,\cdots,\theta_n)=\prod_{j=1}^{n}\Lambda(\theta_j),\quad F_0=1.$$

III. The antiperiodic XXZ model: Bethe states

The Bethe state

$$|\lambda_1,\cdots,\lambda_N\rangle=\prod_{j=1}^{N}D(\lambda_j)|\Omega;\{\theta_j\}\rangle$$

$$\langle\theta_{p_1},\cdots,\theta_{p_n}|\lambda_1,\cdots,\lambda_N\rangle=\left\{\prod_{j=1}^{N}d(\lambda_j)\right\}F_n(\theta_{p_1},\cdots,\theta_{p_n})$$

$$[l]_q=\frac{1-q^{2l}}{1-q^2},\qquad [0]_q=1,$$
$$[l]_q!=[l]_q\,[l-1]_q\cdots[1]_q,\qquad q=e^{\eta},$$
$$\tilde{B}^-=\lim_{u\to+\infty}\left\{\left(2\sinh\eta\,e^{-u}\right)^{N-1}e^{\sum_{l=1}^{N}\theta_l}B(u)\right\}$$

Requirement !

$$\langle\theta_{q_1},\cdots,\theta_{q_n}|\Omega;\{\theta_j\}\rangle=\prod_{l=1}^{n}a(\theta_{p_l})e^{\theta_{p_l}},\quad n=0,\cdots,N.$$

$$|\Omega;\{\theta_j\}\rangle=\sum_{l=0}^{\infty}\frac{\left(\tilde{B}^-\right)^l}{[l]_q!}|0\rangle=\sum_{l=0}^{N}\frac{\left(\tilde{B}^-\right)^l}{[l]_q!}|0\rangle,$$

q-spin coherent state

From known eigenvalue and creation operator to retrieve initial state!

IV. The XYZ model

Baxter 72
Faddeev & Takhtajan **79**

$$H = \frac{1}{2}\sum_{n=1}^{N}(J_x\sigma_n^x\sigma_{n+1}^x + J_y\sigma_n^y\sigma_{n+1}^y + J_z\sigma_n^z\sigma_{n+1}^z)$$

$$J_x = e^{i\pi\eta}\frac{\sigma(\eta+\frac{\tau}{2})}{\sigma(\frac{\tau}{2})}, \quad J_y = e^{i\pi\eta}\frac{\sigma(\eta+\frac{1+\tau}{2})}{\sigma(\frac{1+\tau}{2})}, \quad J_z = \frac{\sigma(\eta+\frac{1}{2})}{\sigma(\frac{1}{2})},$$

$$\theta\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}(u,\tau) = \sum_{m=-\infty}^{\infty}\exp\left\{i\pi\left[(m+a_1)^2\tau + 2(m+a_1)(u+a_2)\right]\right\}$$

$$\sigma(u) = \theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(u,\tau), \quad \zeta(u) = \frac{\partial}{\partial u}\{\ln\sigma(u)\}.$$

III. The XYZ model: Inhomogeneous T-Q

$$\Lambda(u) = e^{2i\pi l_1 u + i\phi} a(u) \frac{Q_1(u-\eta)}{Q_2(u)} + e^{-2i\pi l_1(u+\eta) - i\phi} d(u) \frac{Q_2(u+\eta)}{Q_1(u)}$$
$$+ c\, \frac{\sigma^m\left(u+\frac{\eta}{2}\right) a(u) d(u)}{\sigma^m(\eta) Q_1(u) Q_2(u)}, \qquad (3$$

$$Q_1(u) = \prod_{j=1}^{M} \frac{\sigma(u-\mu_j)}{\sigma(\eta)}, \quad Q_2(u) = \prod_{j=1}^{M} \frac{\sigma(u-\nu_j)}{\sigma(\eta)}.$$

$$N + m = 2M.$$

The minimal T-Q
m=1 for odd N
m=2 for even N

V. Concluding Remarks & Perspective

Intrinsic properties of R matrix

Operator product identities

$$t(\theta_j)t(\theta_j-\eta)=a(\theta_j)d(\theta_j)$$

+ Asymptotic behavior of the polynomial

Inhomogeneous T-Q

$$\Lambda(u)=a(u)\frac{Q_1(u-\eta)}{Q_2(u)}+d(u)\frac{Q_2(u+\eta)}{Q_1(u)}+c(u)\frac{a(u)d(u)}{Q_1(u)Q_2(u)}$$

Regularity → Bethe Ansatz equations

V. Concluding Remarks & Perspective

- Spin torus: [Phys. Rev. Lett. 111, 137201 (2013)]
- Open XXX: [Nucl. Phys. B 875, 152 (2013)]
- Open XXZ & XYZ: [Nucl. Phys. B 877, 152 (2013)]
- Periodic XYZ: [Nucl. Phys. B 886, 185 (2014)]
- Hubbard: [Nucl. Phys. B 879, 98 (2014)]
- t-J: [JSTAT P04031, (2014)]
- SU(n) & nested ODBA [JHEP 04, 143 (2014)]
- Thermodynamics: [Nucl. Phys. B 884, 17 (2014)]
- Izergin-Korepin: [JHEP 06, 128 (2014)]
- Spin-s Heisenberg: [JHEP 02, 036 (2015)]
- Retrieve the eigenstate [Nuclear Physics B 893 , 70 (2015);
  JSTAT P05014, (2015)]

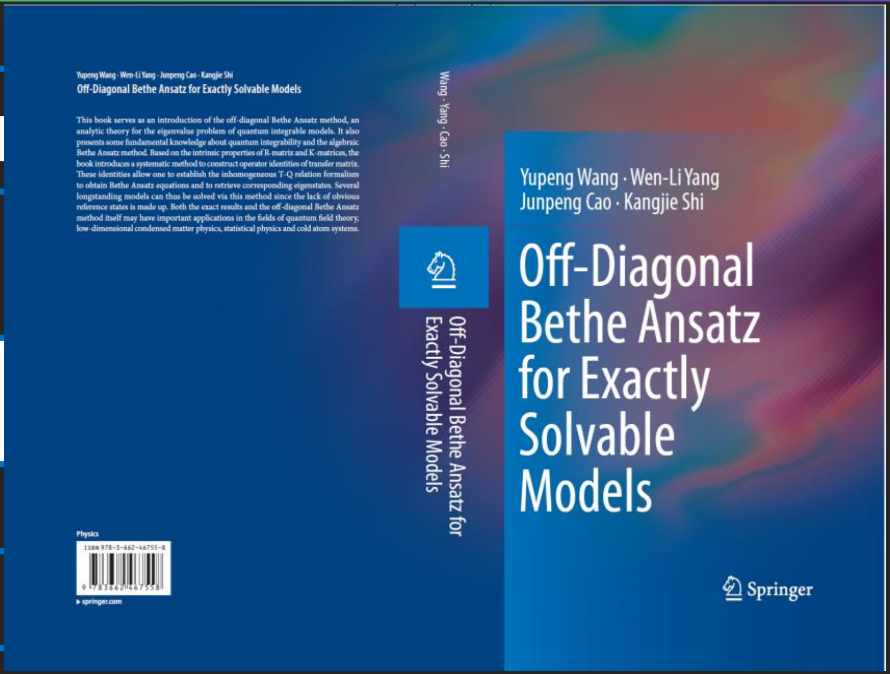
V. Concluding Remarks & Perspective

High rank systems: An, Br

Fusion

$$t^{(1)}(\theta_j)t^{(r)}(\theta_j-\eta)\sim t^{(r+1)}(\theta_j),\quad r=1,\cdots,N-$$

The method works for all bound



ODBA in principle provides a unified method to solve the quantum integrable models

The irreducible inhomogeneous T-Q must imply non-trivial topological nature of the system! Mathematics?


Thanks!