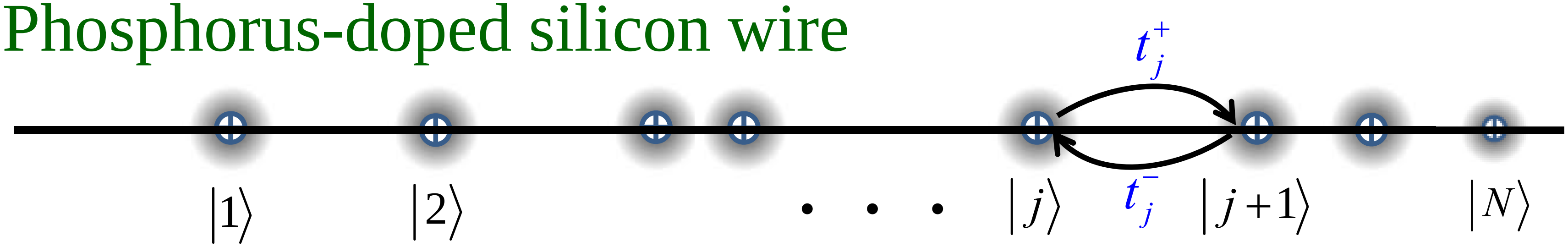
Electron localization in a disordered Phosphorus-doped silicon wire



tight binding model with random hopping disorder

$$H = -\sum_{j=1}^{N} \left[t_j^+ \,|j+1\rangle\langle j| + t_j^- \,|j\rangle\langle j+1| \right]$$

$$t_j^+ = t_j^- \equiv t_j = s_0 + s_j > 0$$

$$s_j \in [-\Delta, \Delta],\ \Delta = 0.5 s_0$$

PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON
Bell Telephone Laboratories, Murray Hill, New Jersey

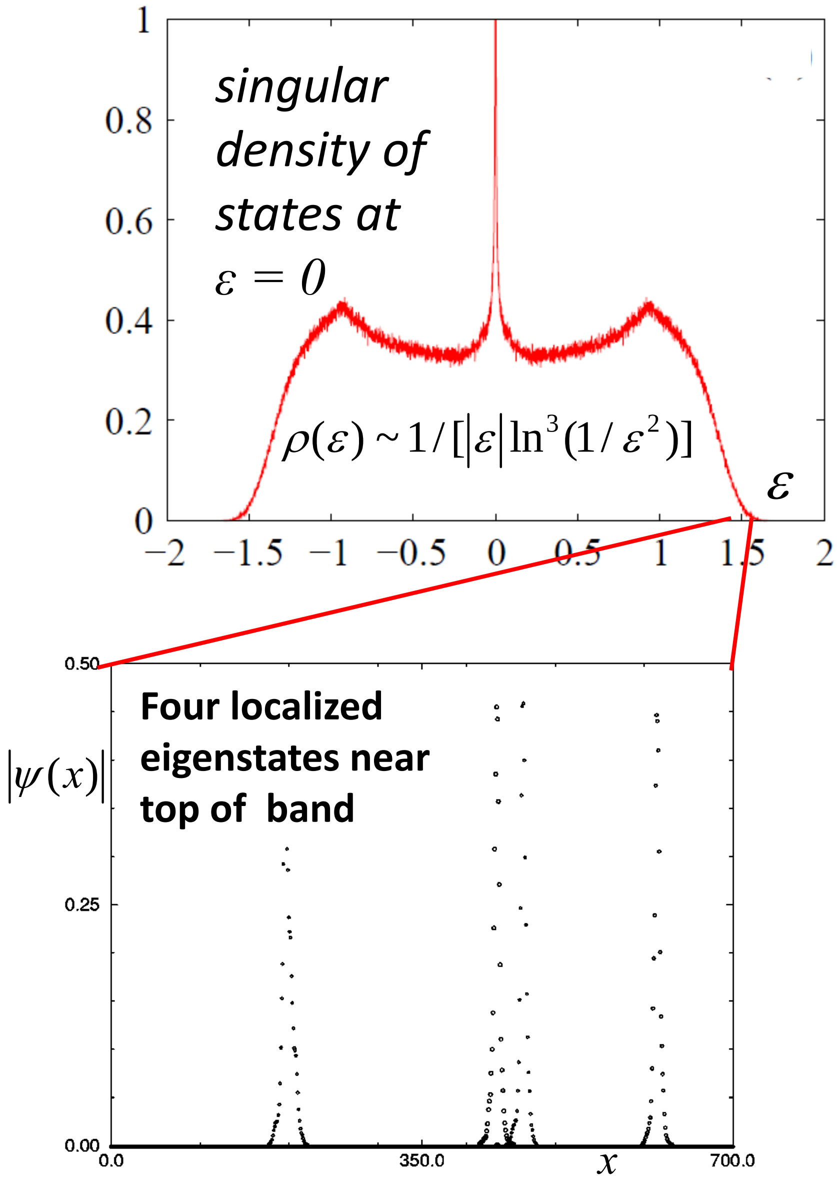
(Received October 10, 1957)

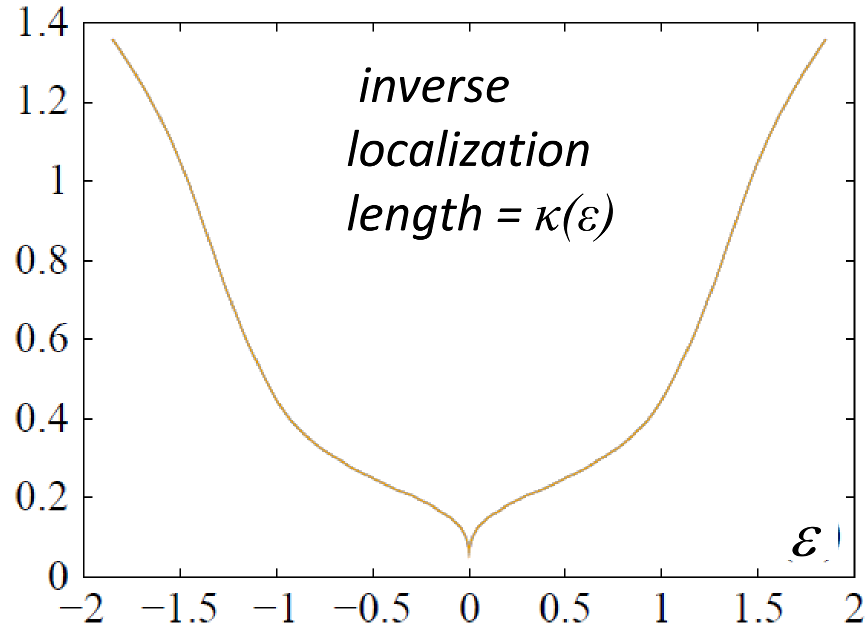
This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

"All states are localized in one dimension…"

$$H = -\sum_{j=1}^{N}\left[t_j \,|j\rangle\langle j+1| + t_j \,|j+1\rangle\langle j| \right]$$

In fact, there is an extended state at the band center…





diverging localization length at $\varepsilon = 0$

$$\lim_{\varepsilon \to 0} \xi_{loc}(\varepsilon) \sim \ln(1/\varepsilon^2) \equiv 1/\kappa(\varepsilon)$$

Variable range hopping in a partially filled band leads to highly non-Ohmic current-voltage relations in one dimension at $T = 0$

$$I \sim I_0 \exp[-(V_0 / V)^{1/2}]$$

<u>Non-Hermitian</u> Localization in Biological Networks

- *Non-Hermitian matrices, with complex eigenvalue spectra, arise naturally in simple models of sparse neural (and ecological) networks.*
- *Striking departures from the conventional wisdom about localization in one-dimensional non-Hermitian random matrices*
- *An intricate eigenvalue spectrum controls the spontaneous activity and induced response. <u>Directed</u> rings of neurons lead to a hole or "band gap" centered on the origin in the complex plane.*
- *All states are extended on the rim of this hole, while the states outside the hole are localized.*

Ariel Amir
Harvard/SEAS

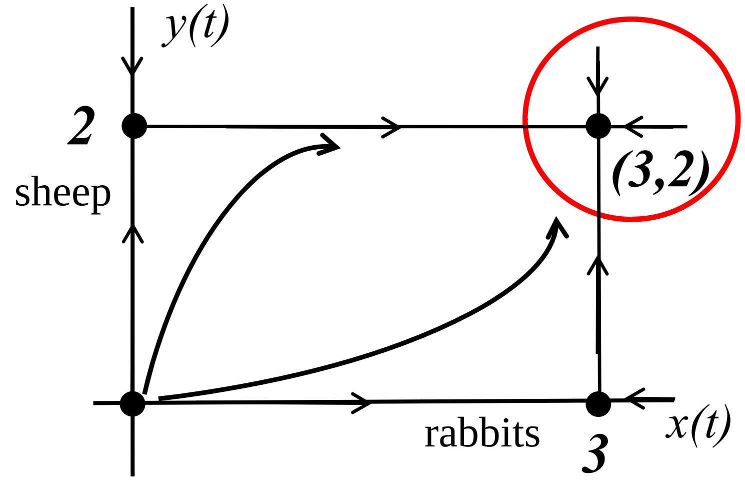
Naomichi Hatano
University of Tokyo

Eigenvectors and eigenvalues in biology: rabbits vs. sheep

$$\frac{dx}{dt} = 3x(1 - x/3)$$

$$\frac{dy}{dt} = 2y(1 - y/2)$$

decoupled model: two logistic equations



x(t) = number of rabbits
y(t) = number of sheep

linearize about the fixed point at (3,2)

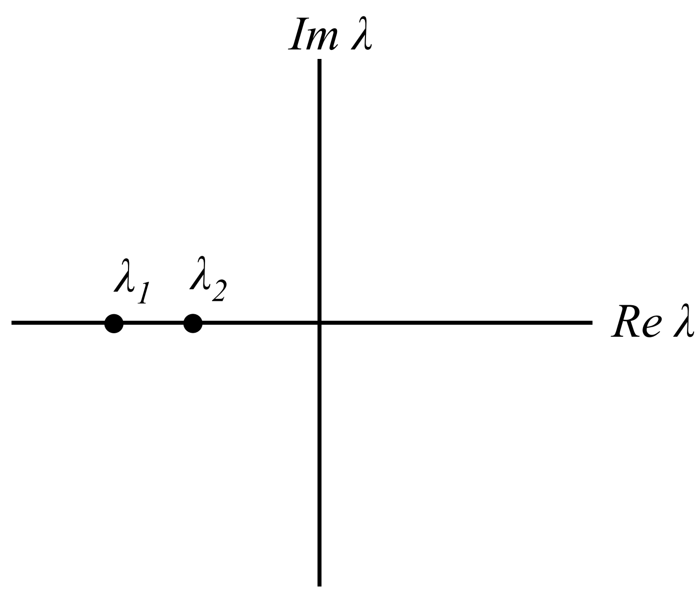
$x'(t) = x(t) - 3,\ \ y'(t) = y(t) - 2$

$$\begin{pmatrix} dx'(t)/dt \\ dy'(t)/dt \end{pmatrix} \approx \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

$x'(t) = x'(0)e^{\lambda_1 t}, y'(t) = y'(0)e^{\lambda_2 t}$

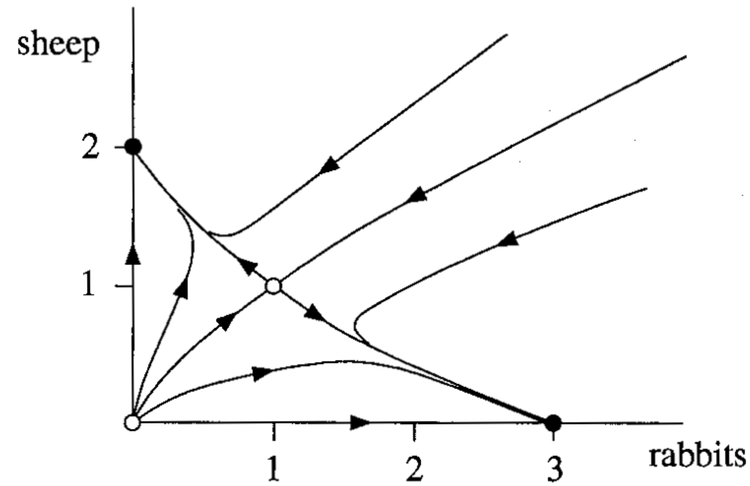
two real eigenvalues:

$\lambda_1 = -3,\ \lambda_2 = -2,$ stable fixed point



Eigenvectors and eigenvalues in biology: rabbits vs. sheep

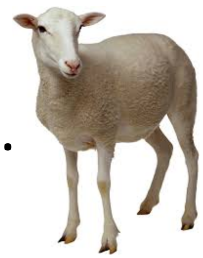
competitive exclusion



$$\frac{dx}{dt} = 3x(1 - x/3 - 2y/3)$$

$$\frac{dy}{dt} = 2y(1 - y/2 - x/2)$$

vs.



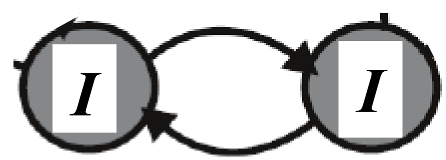
or…. two coupled inhibitory neurons

$v_i =$ firing rate deviation from the background rate of the ith neuron

$$\tau \frac{dv_1}{dt} = -v_1 + \tanh\left[M_{11}v_1 + M_{12}v_2\right]$$

$$\tau \frac{dv_2}{dt} = -v_2 + \tanh\left[M_{21}v_1 + M_{22}v_2\right]$$

$$\mathrm{M} = \begin{pmatrix} 0 & -s \\ -s & 0 \end{pmatrix}$$

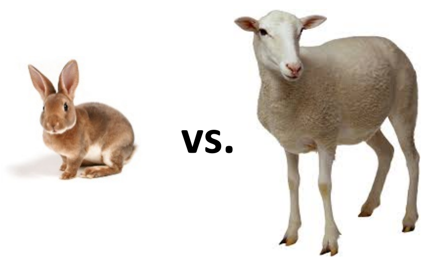


S. H. Strogatz, Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering. Westview press, 2014.

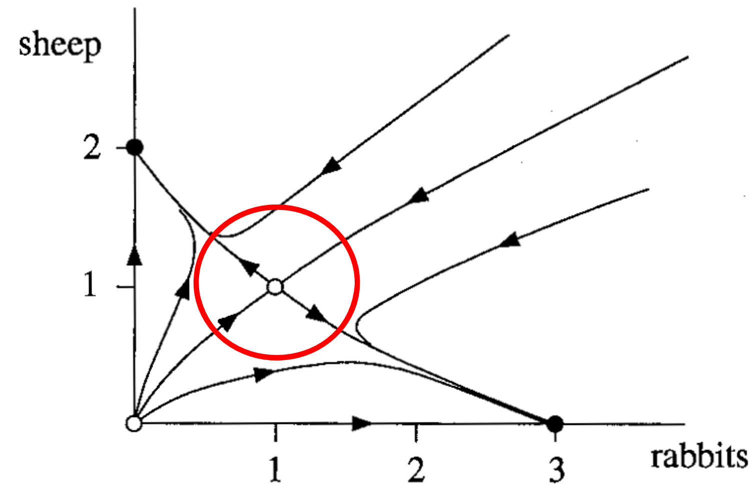
Eigenvectors and eigenvalues in biology: Rabbits vs. Sheep

$$\frac{dx}{dt} = 3x(1 - x/3 - 2y/3)$$

$$\frac{dy}{dt} = 2y(1 - y/2 - x/2)$$



either sheep or rabbits win or "fix" at long times...



Four fixed points are obtained: (0,0), (0,2), (3,0), and (1,1).

linearize about the fixed point at (1,1)

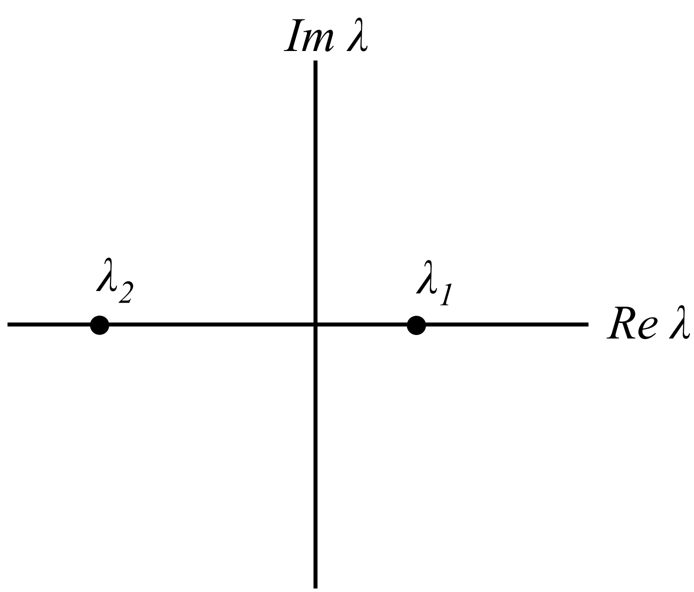
$x'(t) = x(t) - 1, \;\; y'(t) = y(t) - 1$

$$\begin{pmatrix} dx'(t)/dt \\ dy'(t)/dt \end{pmatrix} \approx \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

two real eigenvalues control dynamics:

$\lambda_1 = -1 + \sqrt{2}, \;\; \lambda_2 = -1 - \sqrt{2}$

due to interactions, there is now one stable and one unstable eigendirection



S. H. Strogatz, Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering. Westview press, 2014.

Rabbits vs. Foxes: complex eigenvalues lead to oscillations…

$Y_1(t) =$ number of rabbits

$Y_2(t) =$ number of foxes

$X =$ const. density of grass

II. LOTKA-VOLTERRA EQUATION

$$\frac{dY_1}{dt} = c_1 X Y_1 - c_2 Y_1 Y_2$$
$$\frac{dY_2}{dt} = c_2 Y_1 Y_2 - c_3 Y_2$$

→ 2 fixed point: $(0,0) \quad (\frac{c_3}{c_2}, \frac{c_1 X}{c_2})$

Stability matrix: $M(Y_1, Y_2) = \begin{pmatrix} c_1 X - c_2 Y_2 & -c_2 Y_1 \\ c_2 Y_2 & c_2 Y_1 - c_3 \end{pmatrix}$

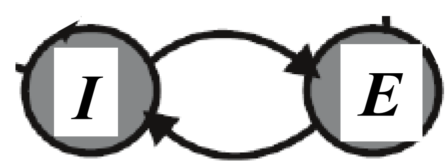
1st fixed point: $M(0,0) = \begin{pmatrix} c_1 X & 0 \\ 0 & -c_3 \end{pmatrix}$

→ eigenvalues are $c_1 X, \ -c_3$ → **Saddle Point**

2nd fixed point: $M(\frac{c_3}{c_2}, \frac{c_1 X}{c_2}) = \begin{pmatrix} 0 & -c_3 \\ c_1 X & 0 \end{pmatrix}$

→ eigenvalues are $\pm i\sqrt{c_3 c_1 X}$ → **Center**

or…. coupled excitatory & inhibitory neurons



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$$\frac{dY_1}{dt} = c_1 X Y_1 - c_2 Y_1 Y_2$$

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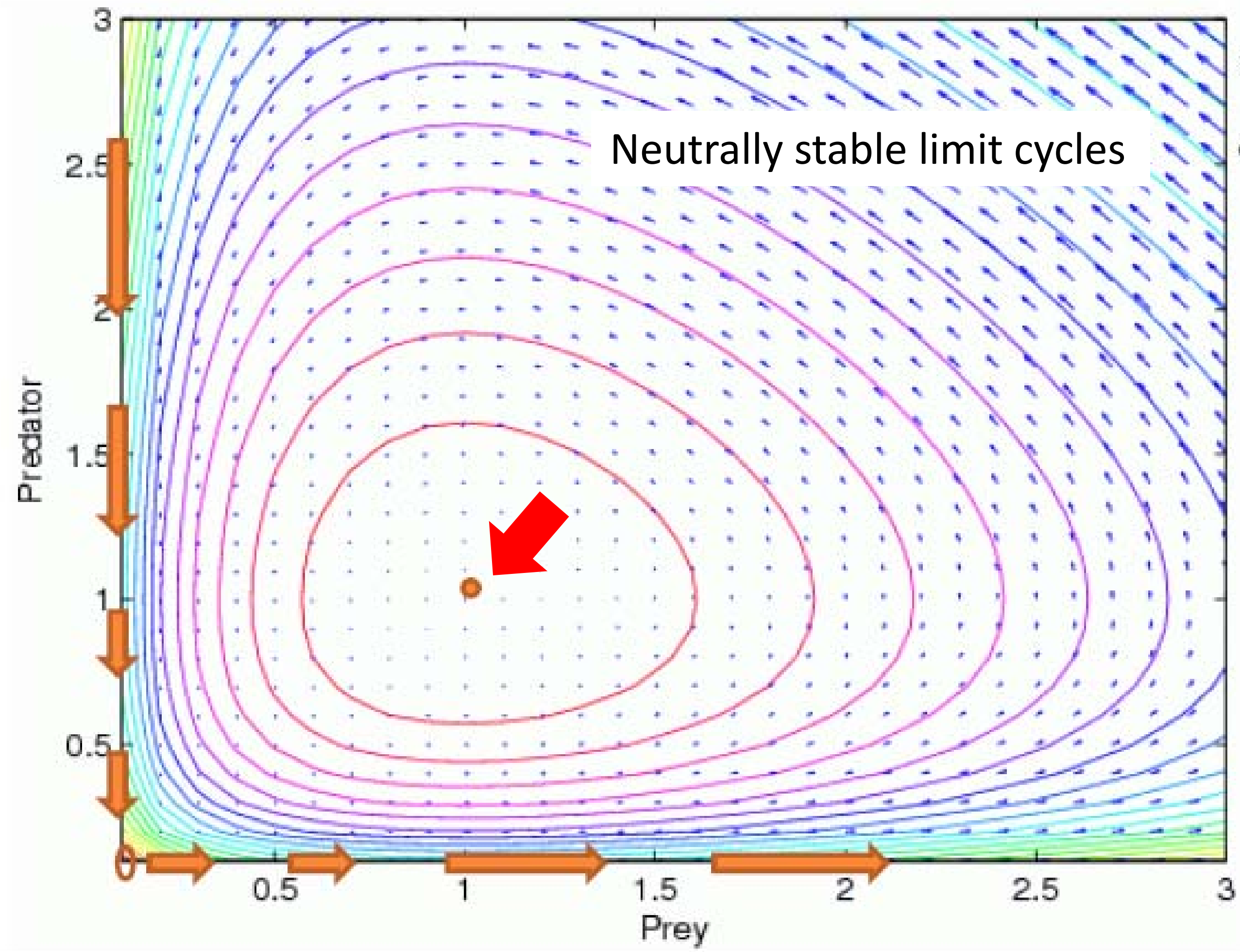
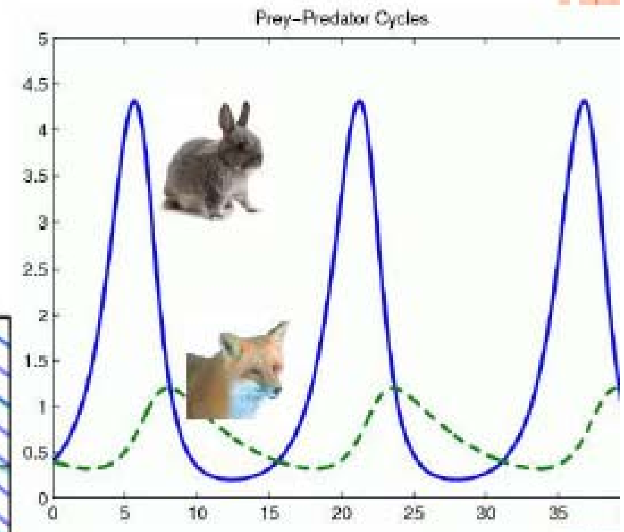
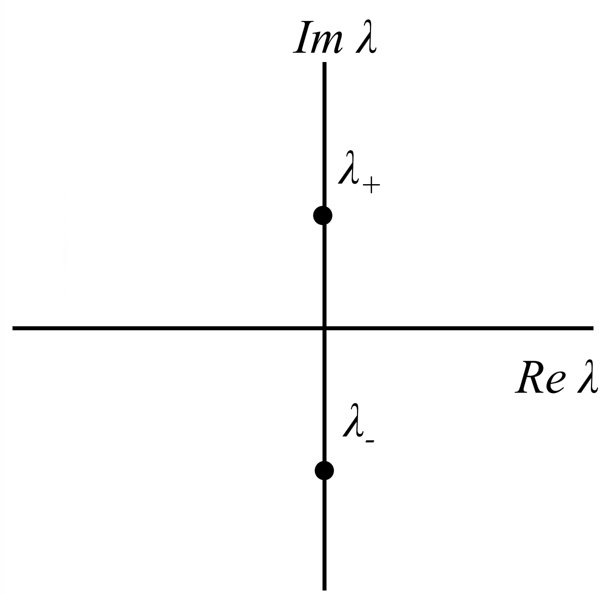






$$\lambda_{\pm} = \pm i\sqrt{c_3 c_1 X}$$

Random matrix theory applied to N-species ecology models (N >> 1)

1. Assume each species in isolation would obey a stable logistic equation with stable eigenvalue -1 then switch on random interactions of either sign

$$\frac{dx_i}{dt} = x_i(1-x_i) - \sum_{j\neq 1}^{N} B_{ij}x_i x_j; \qquad \text{Let } x_i'(t) = x_i(t) - x^* = x - 1$$

R. M. May. Nature, 238 413 (1972)

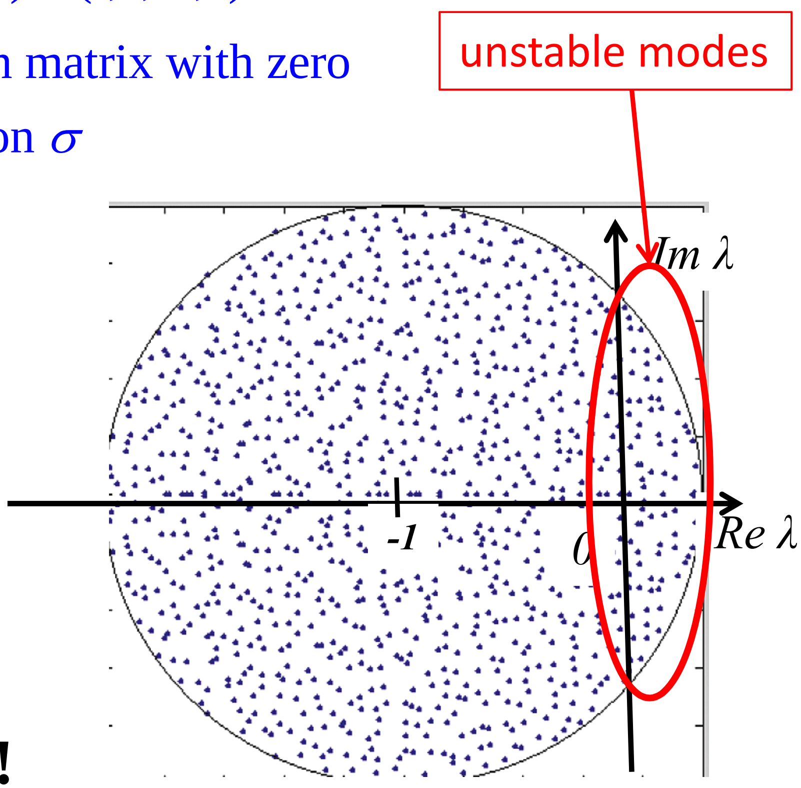
2. $\dfrac{dx_i'(t)}{dt} \approx \sum_{j=1}^{N} A_{ij} x_j'(t)$, $\vec{x}'(t)$ is an N-component vector of species deviations from the logistic fixed point $(x_1^*, x_2^*, \ldots, x_N^*) = (1, 1, \ldots, 1)$

3. $\overleftrightarrow{A} \approx -\overleftrightarrow{I} - \overleftrightarrow{C}$, where $\overleftrightarrow{C}$ is an N-component interaction matrix with zero mean for each element and each with standard deviation σ

The spectrum of $\overleftrightarrow{C}$ is a uniform distribution of complex eigenvalues in unit circle in the complex plane of radius $\sigma\sqrt{N}$.

Universal density of states for large N!

"Girko's Law"

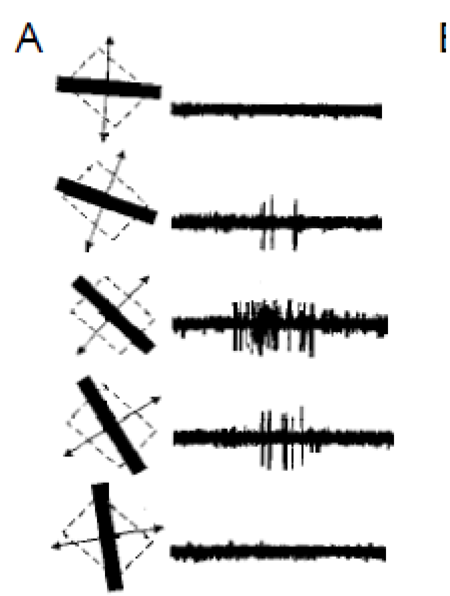


Any ecological system becomes unstable for sufficiently large *N*!

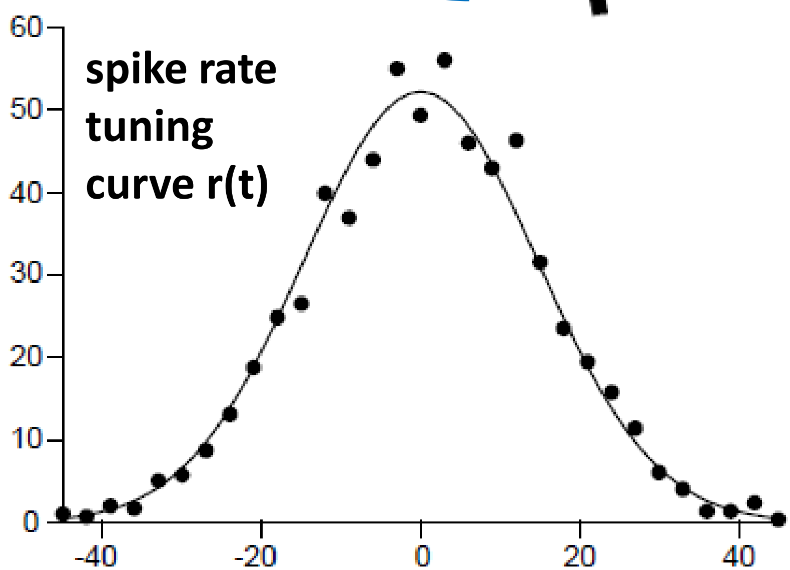
Random Matrices in Neuroscience

Visual stimulus s(t) transferred from retinal neurons → LGN → V1 region of the visual coretex

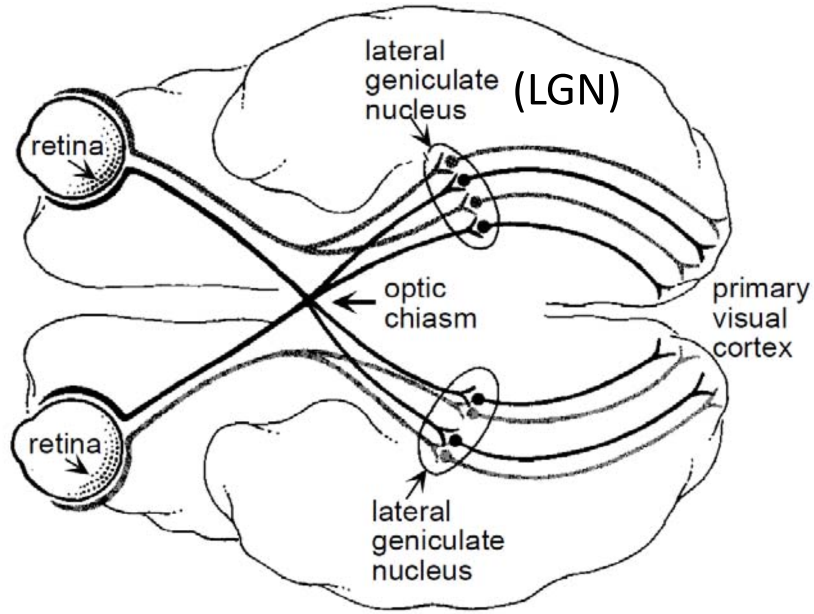
Spike rate r(t) depends on orientation of bar moving across the visual field



spike rate tuning curve r(t)



signal s(t): orientation in degrees



Pathway from the retina through the lateral geniculate nucleus (LGN) to the primary visual cortex

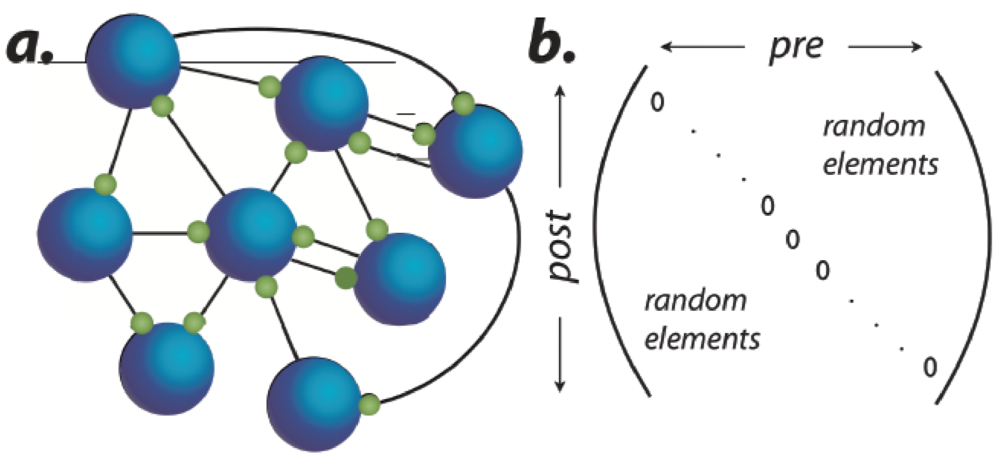
Dayan and Abbott: Theoretical Neuroscience

Random matrix models of the brain (H. Sompolinsky, L. Abbott et alia)

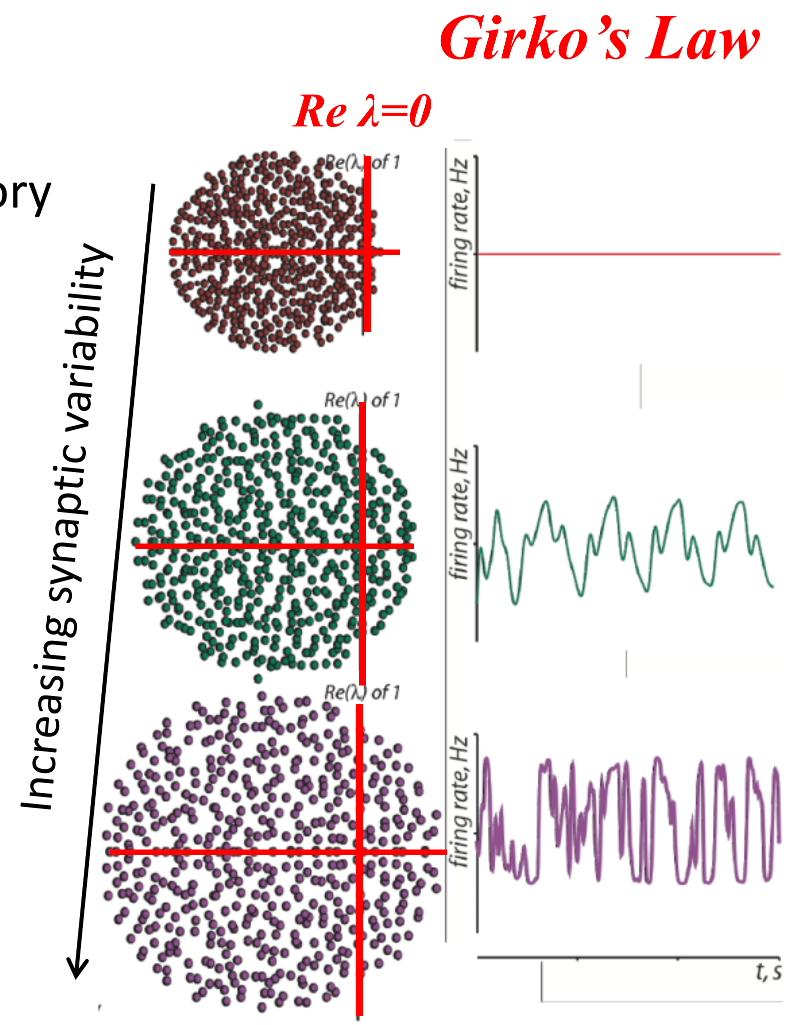
- Random neural connections can be formed during development, with many stochastic attachments of axons and dendrites to other neurons.

- Over time, pruning and strengthening/weakening of connections allow neural circuits to "learn" various functions.

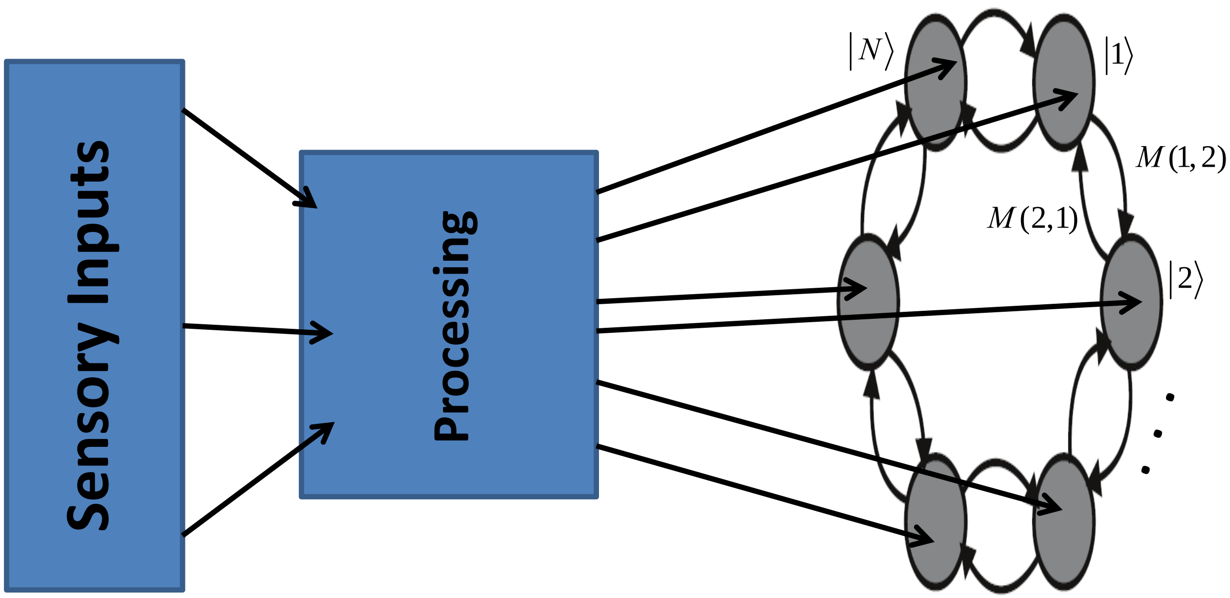
- The spectra and eigenfunctions of completely random neural networks with a mixture of inhibitory and excitatory connections, can describe neural activity during the early stages of development.



K. Rajan, 2009 Spontaneous and Stimulus-driven Network Dynamics. Doctoral Dissertation, Columbia University.



Random matrix model of a sparse neural network



Sensory inputs, possibly after a processing step, are sent via feed forward couplings into a circular ring of N neurons Note that M(1,2) and M(2,1) can not only be unequal, but also of opposite sign, if one direction is excitatory and the other inhibitory.

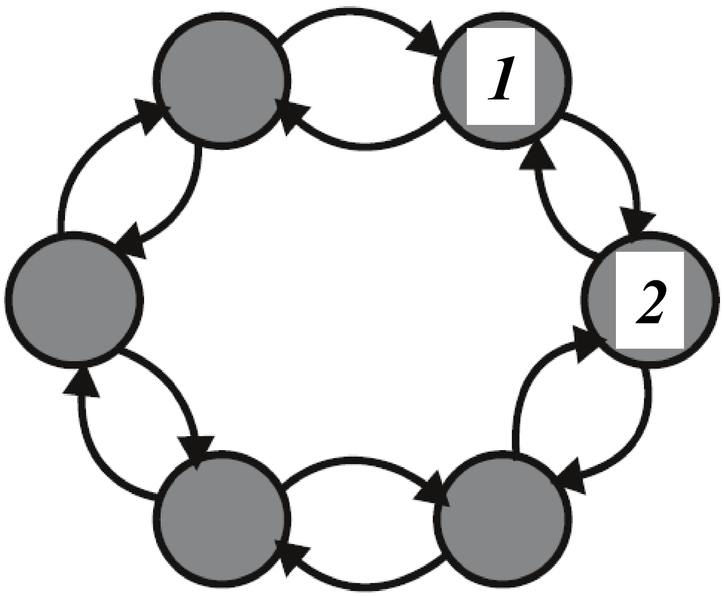
$v_i =$ firing rate deviation from background of the ith neuron in recurrent network

$u_j =$ input firing rate of the j^{th} neuron in the input (feed forward) network

$$\tau \frac{dv_i}{dt} = -v_i + \tanh\left[\sum_{j=1}^{N} M_{ij} v_j + h_i\right], \quad h_i = \sum_{j=1}^{N} W_{ij} u_j$$

$$\tau \frac{dv_i}{dt} \approx -v_i + \sum_{j=1}^{N} M_{ij} v_j + h_i \quad \text{(linear approximation)}$$

Non-Hermitian neural networks with random excitatory ($M(i,j) > 0$) and inhibitory ($M(i,j) < 0$) connections



$$M = \sum_{j=1}^{N}\left[s_j^+ e^g \,|j\rangle\langle j+1| + s_j^- e^{-g}\,|j+1\rangle\langle j| \right]$$

g provides a systematic clockwise (g > 0) or counterclockwise (g < 0) directional bias

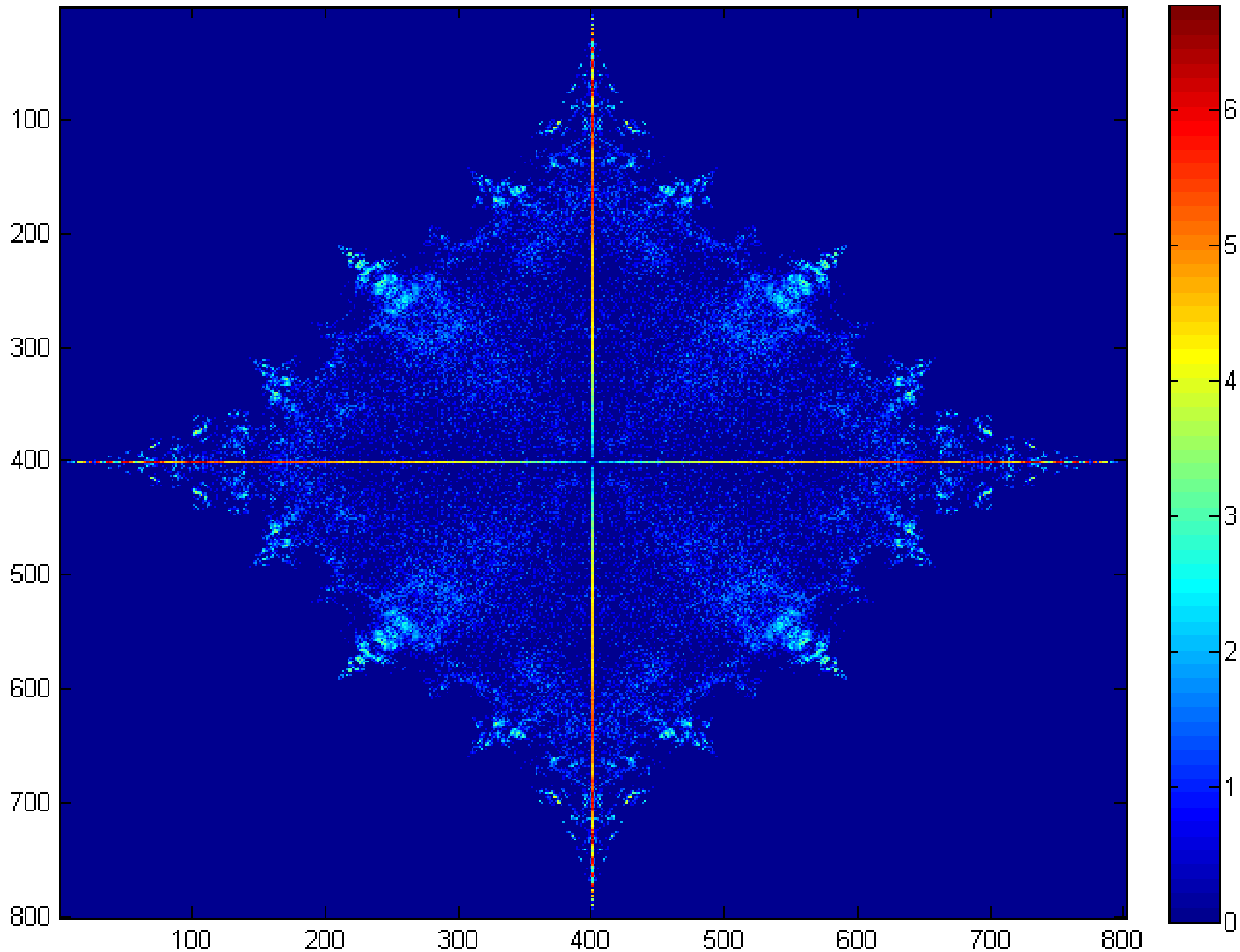
Study eigenvalues and eigenvectors of directed, banded non-Hermitian random matrices

$s_j^+,\ s_j^- = \pm 1,$ indep. random variables;

$$M = \begin{pmatrix} 0 & s_1^+ e^g & 0 & \cdots & s_N^- e^{-g} \\ s_1^- e^{-g} & 0 & s_2^+ e^g & 0 & \\ 0 & s_2^- e^{-g} & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & & s_{N-1}^+ e^g \\ s_N^+ e^g & \cdots & 0 & s_{N-1}^- e^{-g} & 0 \end{pmatrix}$$

Set g = 0 for now → random sign model of J. Feinberg and A. Zee, PRE 59 6433 (1999)

Eigenvalue distribution in the complex plane $\lambda = \lambda_1 + i\lambda_2$



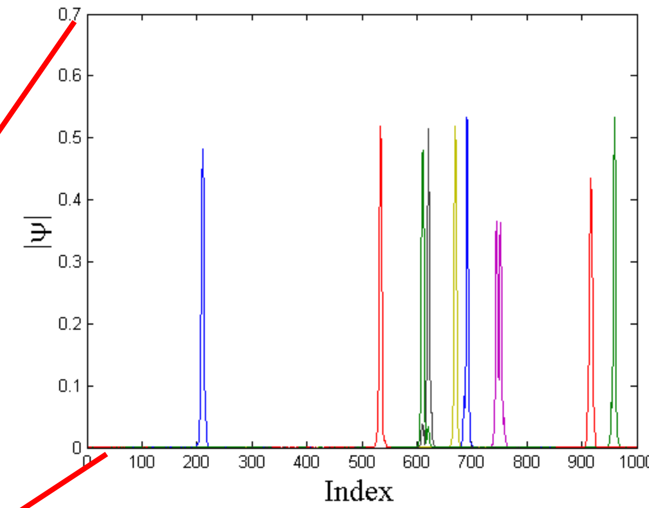
Result of exact diagonalization of 10,000 N x N matrices with N = 5000 and g = 0

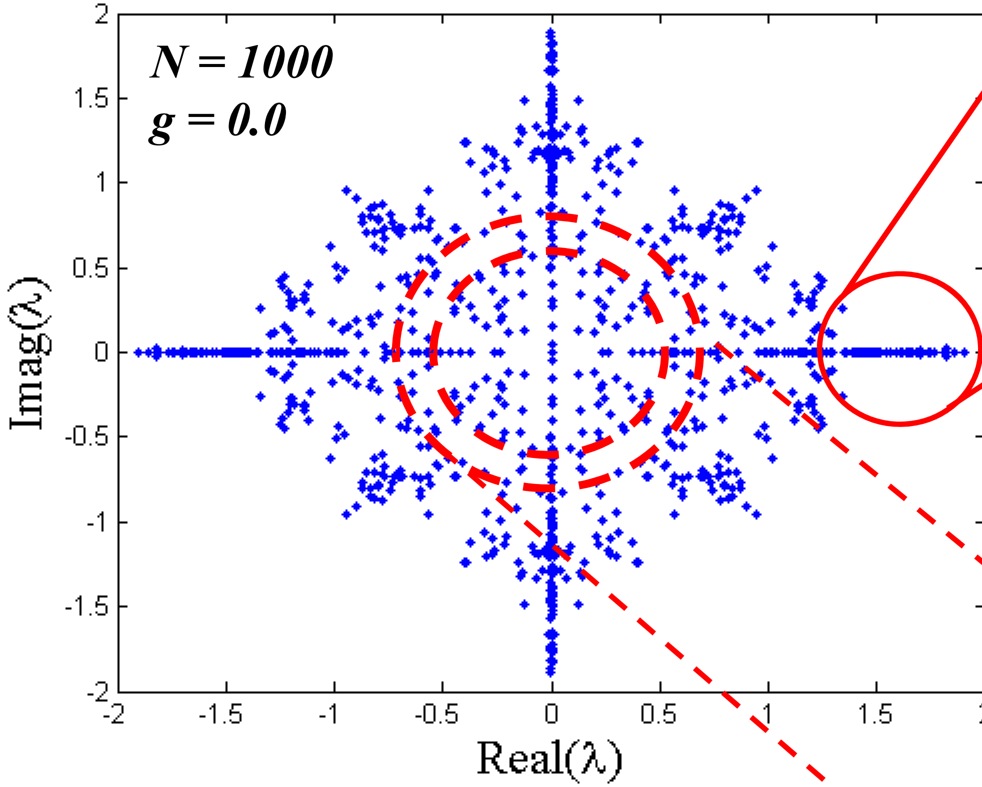
(Suppress negative relaxation rate: Spectrum now shifted to be centered on origin….)

How localized are the eigenfunctions??

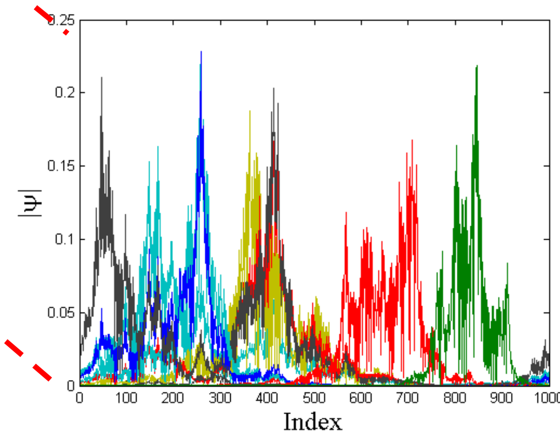
What does “localization” mean?

Eigenfunctions within circle on right side are highly localized w/real eigenvalues





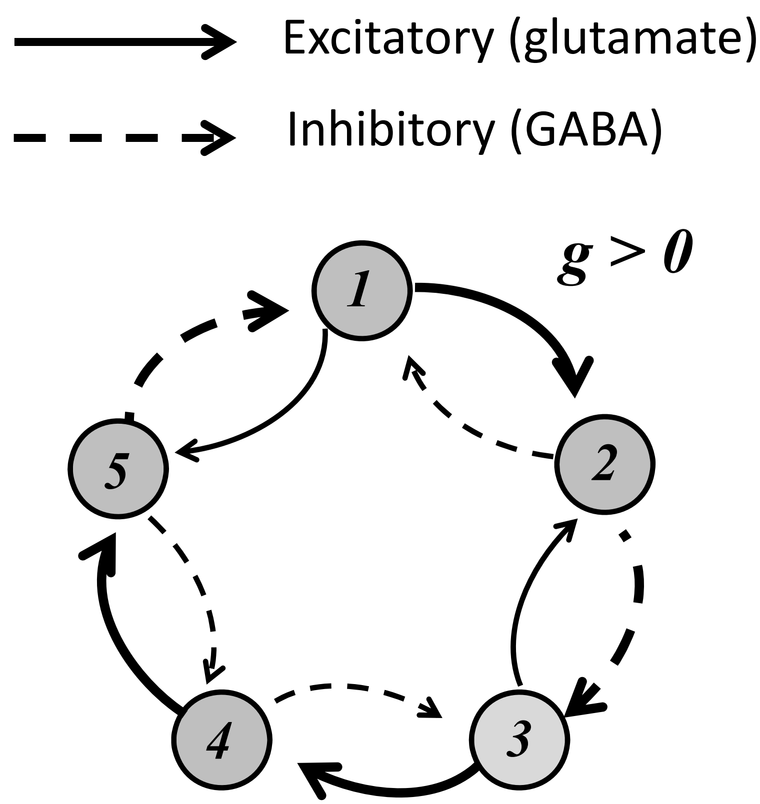
Eigenfunctions in an annulus closer to the origin are more extended



localization length diverges near the origin

$$\lambda = \lambda_1 + i\lambda_2, \quad \xi(\lambda_1, \lambda_2) \sim \frac{1}{(|\lambda_1| + |\lambda_2|)\sqrt{\lambda_1^2 + \lambda_2^2}}$$

What is the effect of the bias parameter g?

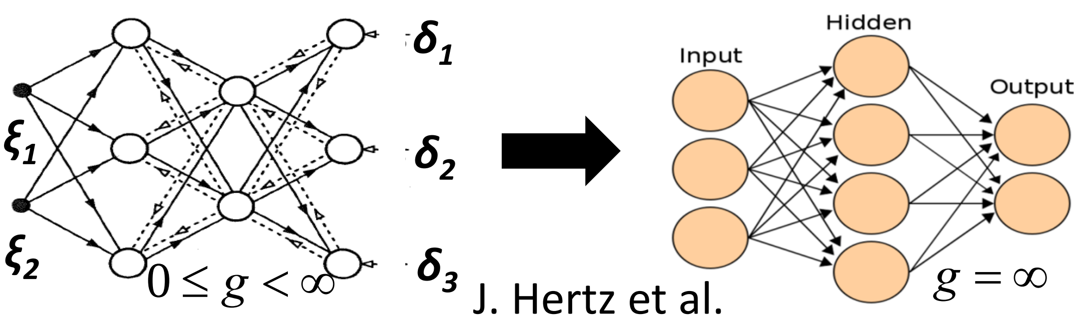


$$M = \begin{pmatrix} 0 & s_1^+ e^{g} & 0 & \dots & s_N^- e^{-g} \\ s_1^- e^{-g} & 0 & s_2^+ e^{g} & 0 & \\ 0 & s_2^- e^{-g} & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & & s_{N-1}^+ e^{g} \\ s_N^+ e^{g} & \dots & 0 & s_{N-1}^- e^{-g} & 0 \end{pmatrix}$$

$s_j^+ = \pm 1, \quad s_j^- = \pm 1$ with equal probability

$0 \le g < \infty$ (no Dale's law for now)

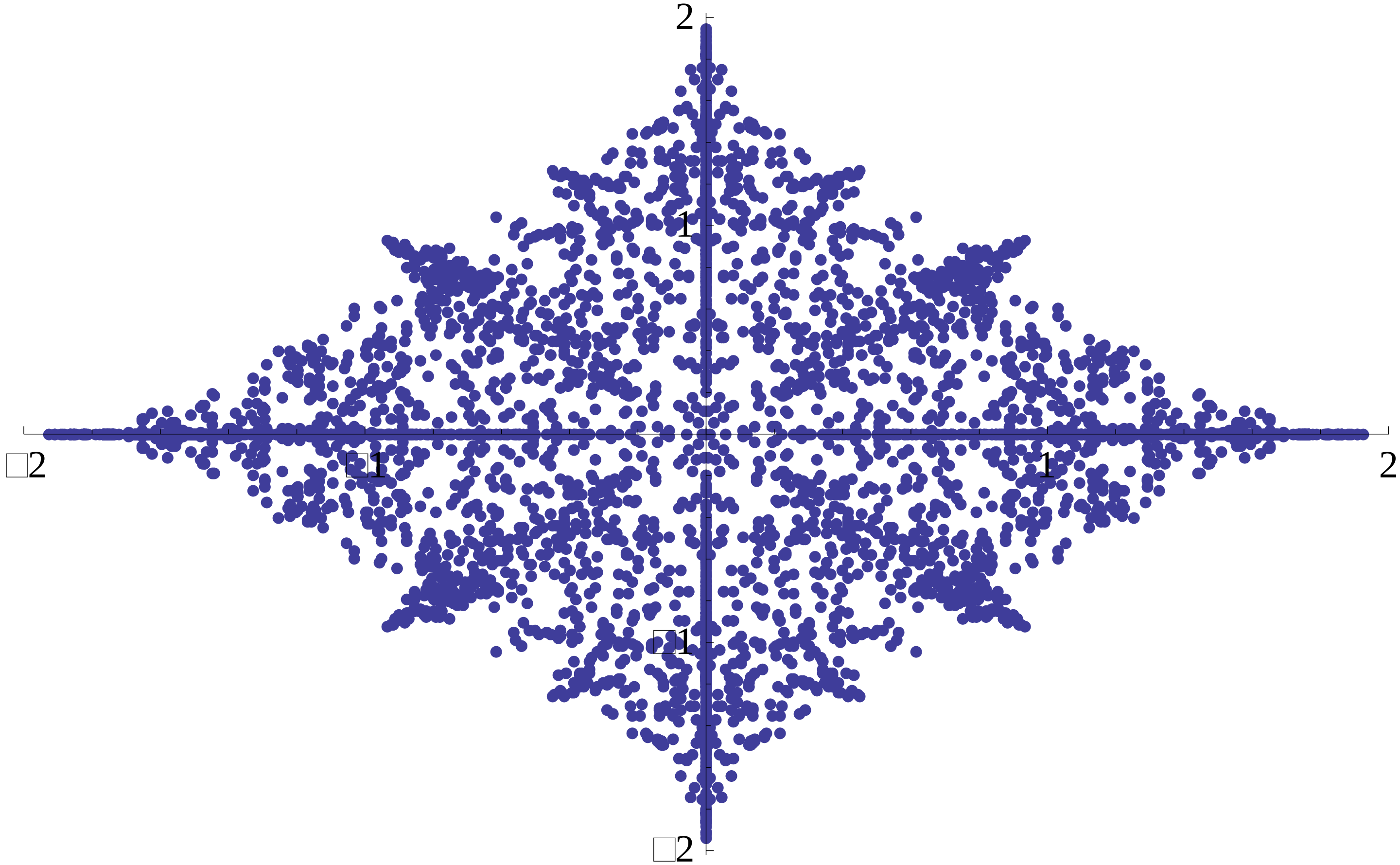
As g increases from 0, it tunes down the amount of feedback in a "feed clockwise" recurrent network…



Similar layered neural nets used for image & sound classification, etc. in machine learning algorithms.
Many layers → "deep learning"

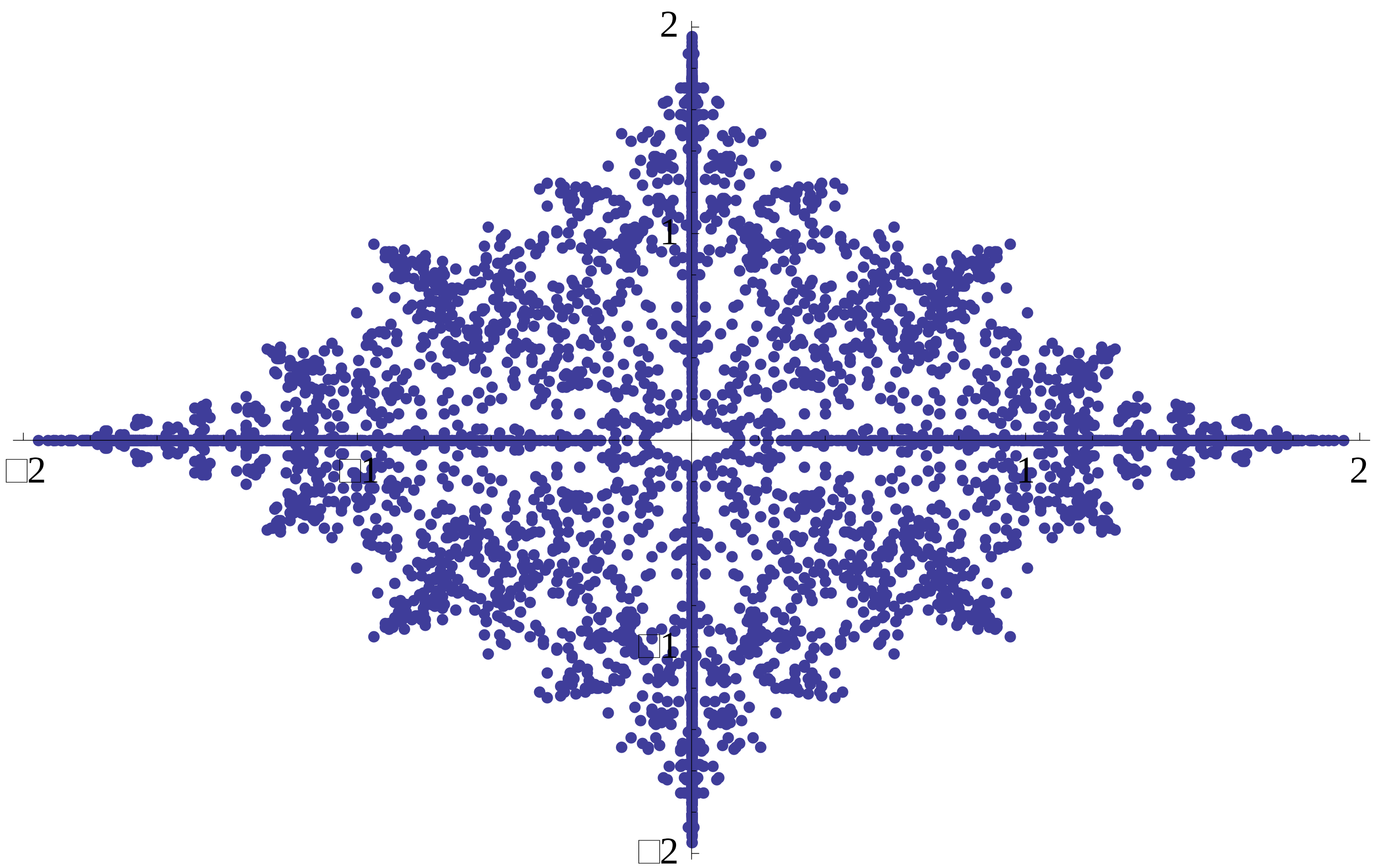
Effect of a directional bias around the chain (g > 0)

N = 5000,   g = 0.0



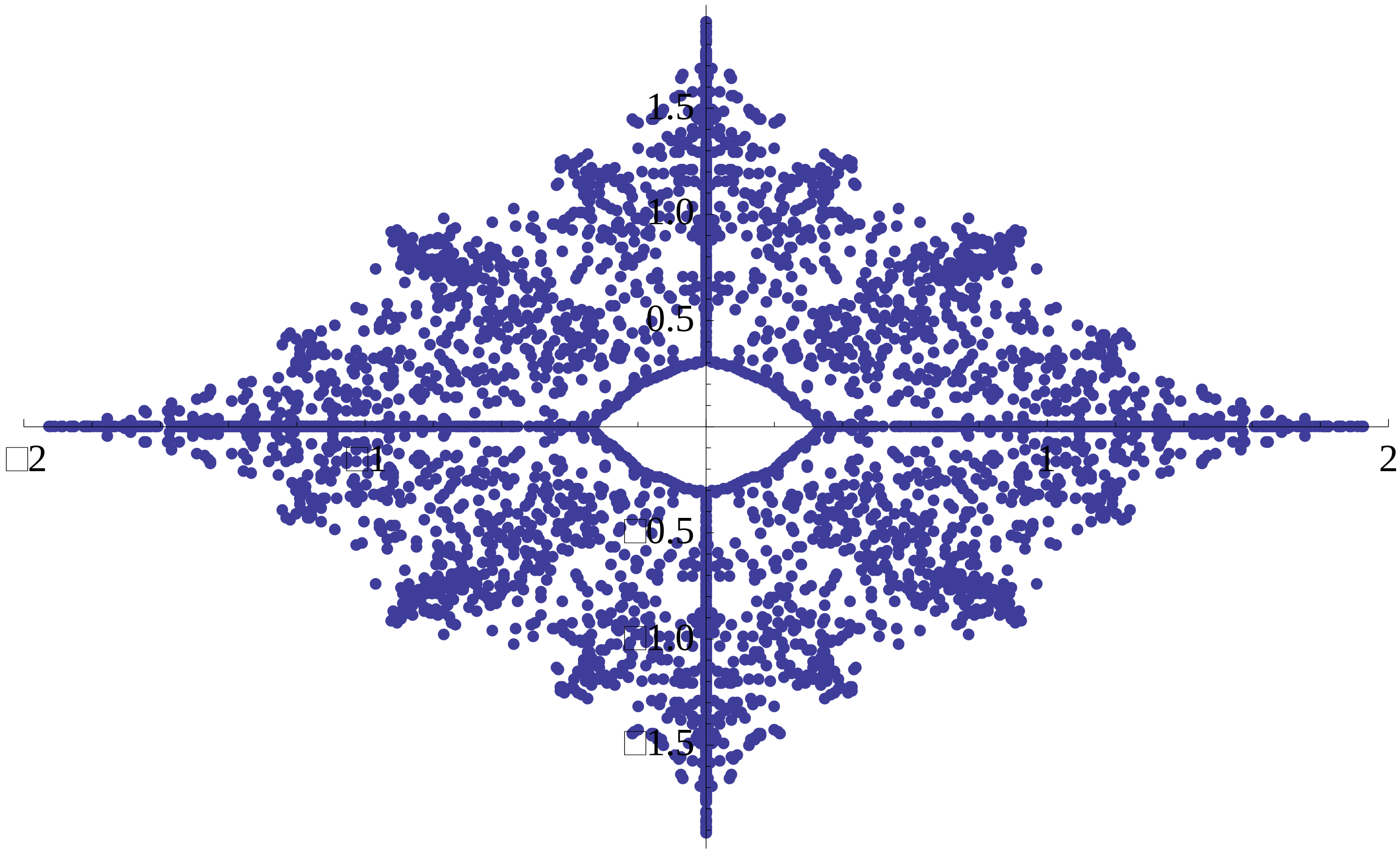
Effect of a directional bias around the chain (g > 0)

N = 5000, g = 0.002



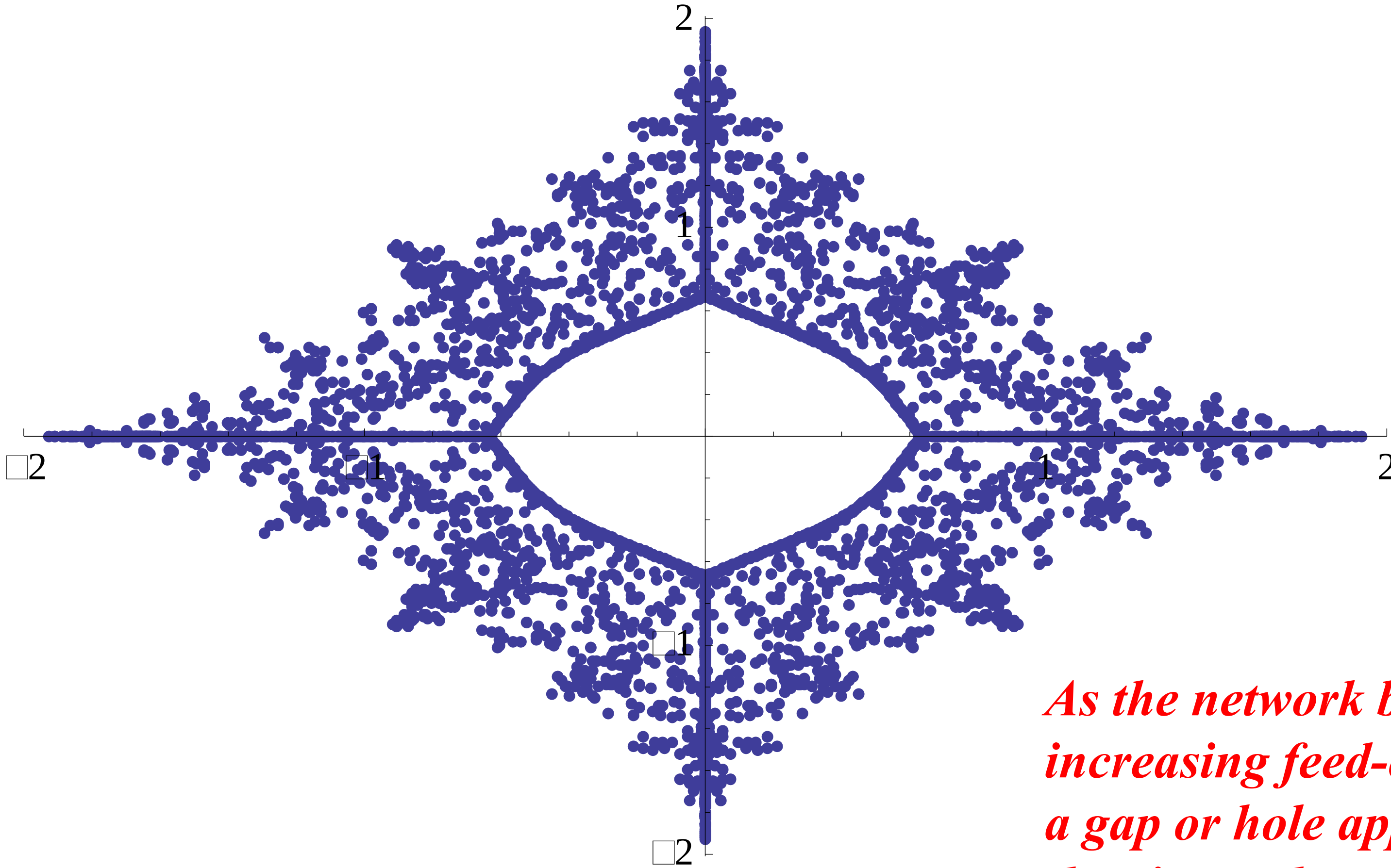
Effect of a directional bias around the chain (g > 0)

N = 5000, g = 0.01



Effect of a directional bias around the chain (g > 0)

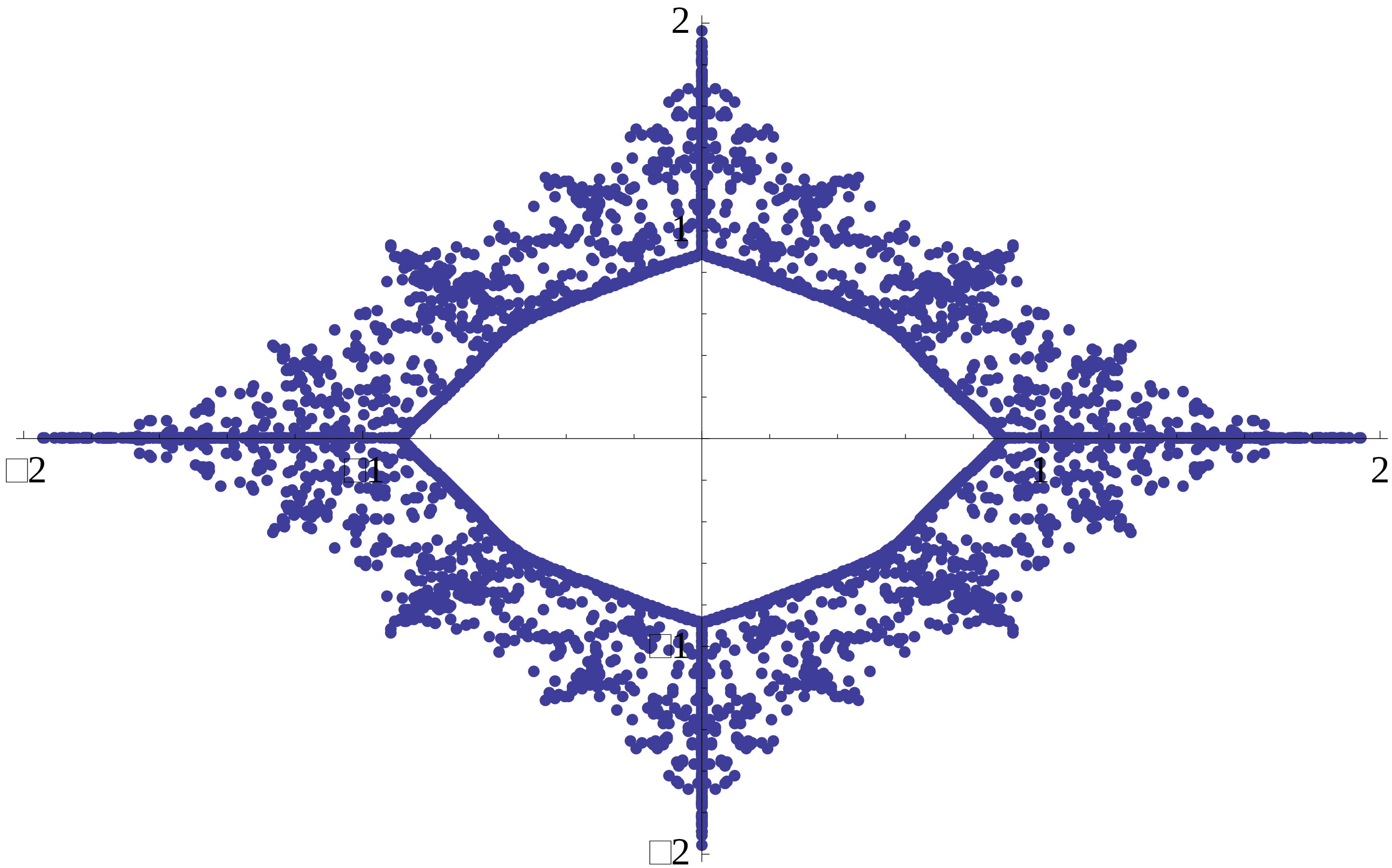
N = 5000, g = 0.05



As the network becomes increasing feed-clockwise, a gap or hole appears in the eigenvalue spectrum in the complex plane…

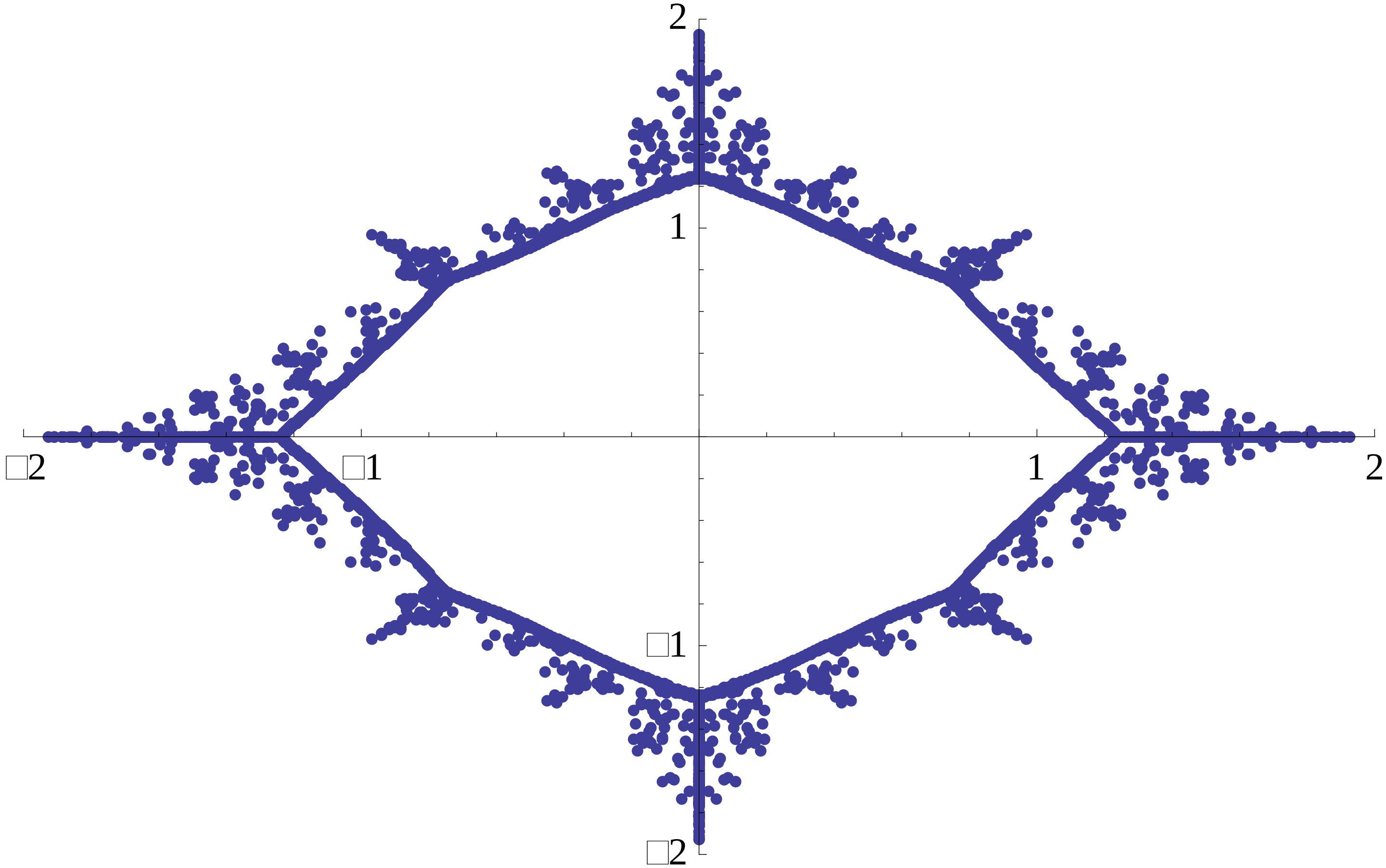
Effect of a directional bias around the chain (g > 0)

N = 5000,   g = 0.1



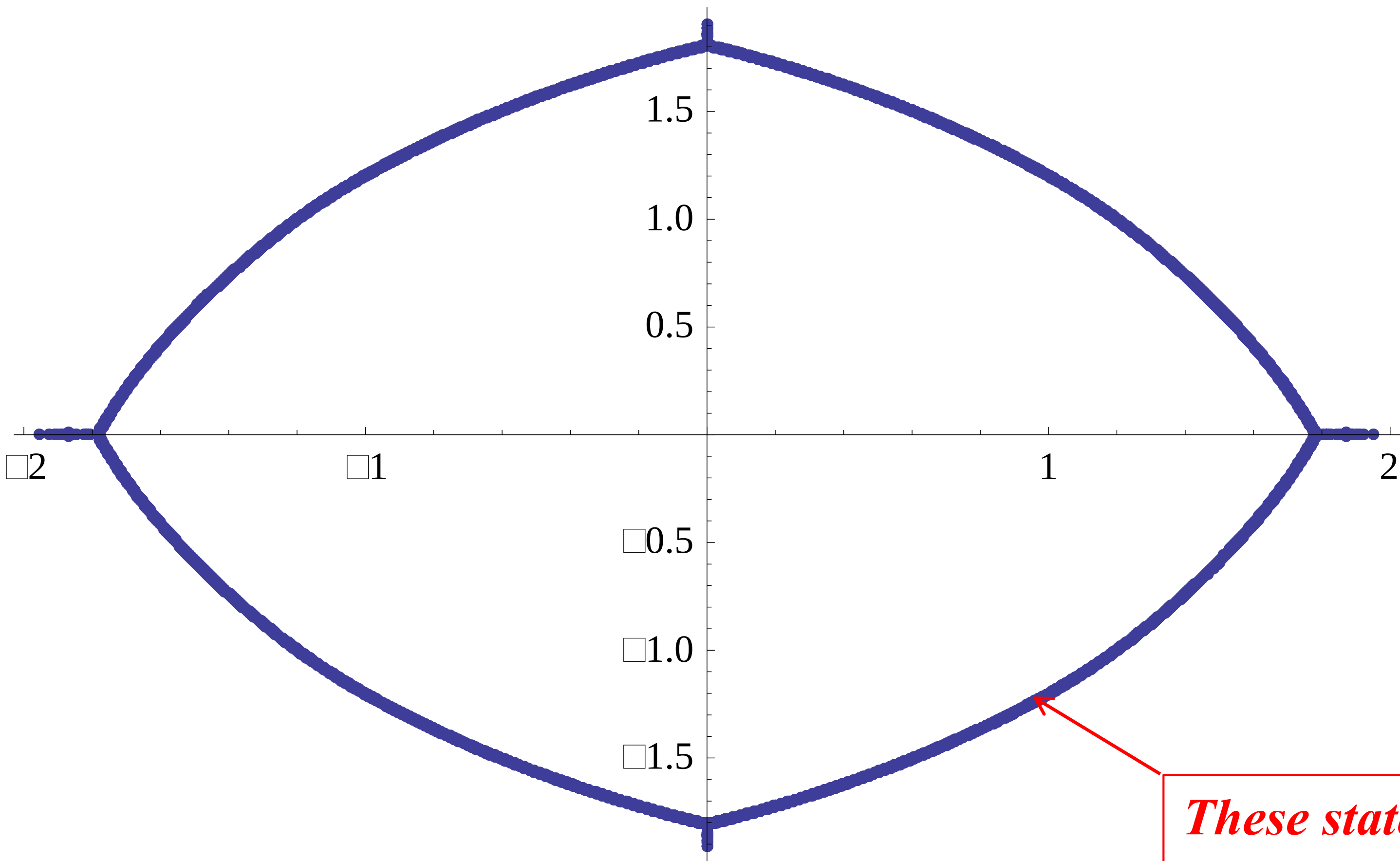
Effect of a directional bias around the chain (g > 0)

N = 5000, g = 0.2



Effect of a directional bias around the chain (g > 0)

N = 5000, g = 0.5



These states move around the ring – they delocalize…

Localization lengths and edge states

Define inverse participation ratio

$$IPR \equiv \sum_j |\phi_j|^4 / \sum_j |\phi_j|^2$$

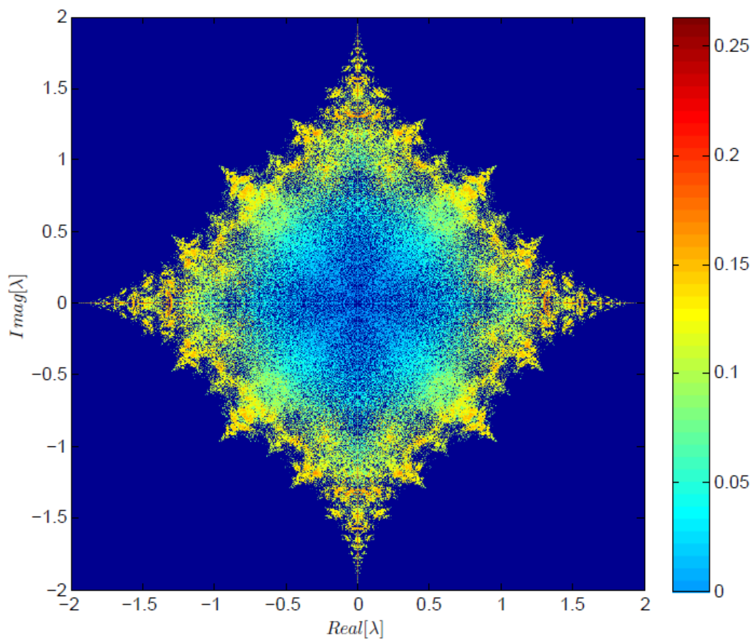
$IPR \sim$ inverse localization length

extended state: $\phi_j \sim 1/\sqrt{N}, \forall j$

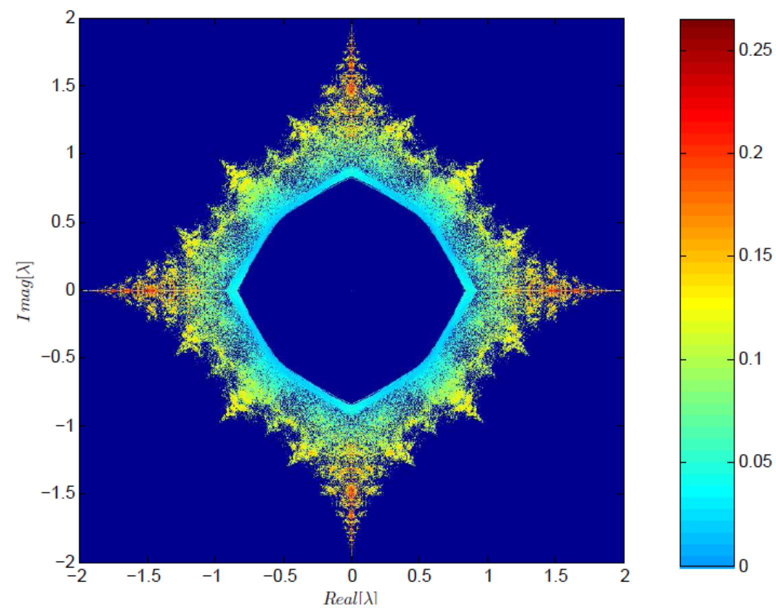
$$IPR \equiv \sum_j |\phi_j|^4 / \sum_j |\phi_j|^2 \sim 1/N << 1$$

localized state, $\phi_j \sim \exp[-|x_j - x_0|/\xi_{loc}]$

$$IPR \equiv \sum_j |\phi_j|^4 / \sum_j |\phi_j|^2 = O(1)$$



Eigenvalue spectrum for g = 0 (or, for <u>any</u> g with open boundary conditions!)



Eigenvalue spectrum for g = 0.1 with periodic boundary conditions

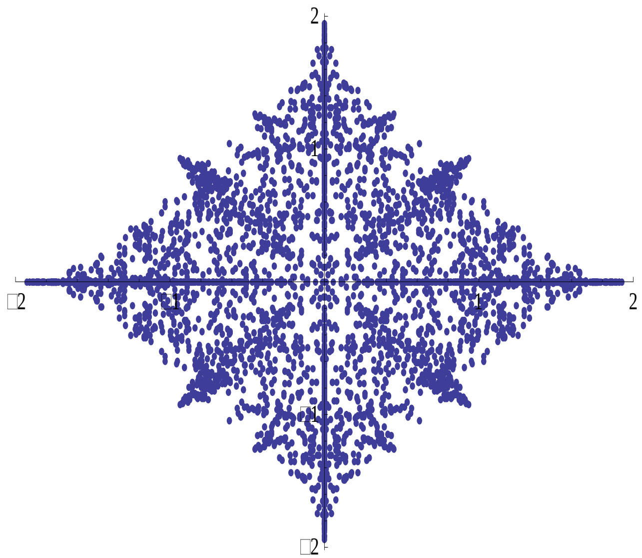
➢ ***Many extended states…. Localization length diverges on the rim of the hole when g > 0 → extended states***

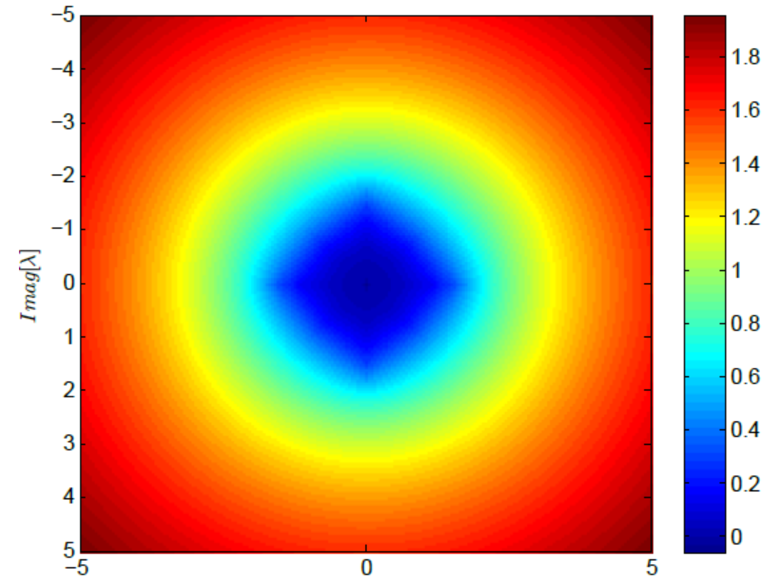
Hole states become edge states for open boundary conditions when g > 0

The shape of the hole

Contours of constant inverse localization length $\kappa(E_1, E_2)$ obtained by solving the electrostatic "Thouless relation"

$$\nabla^2 \kappa(E_1, E_2) = \Phi(E_1, E_2)$$

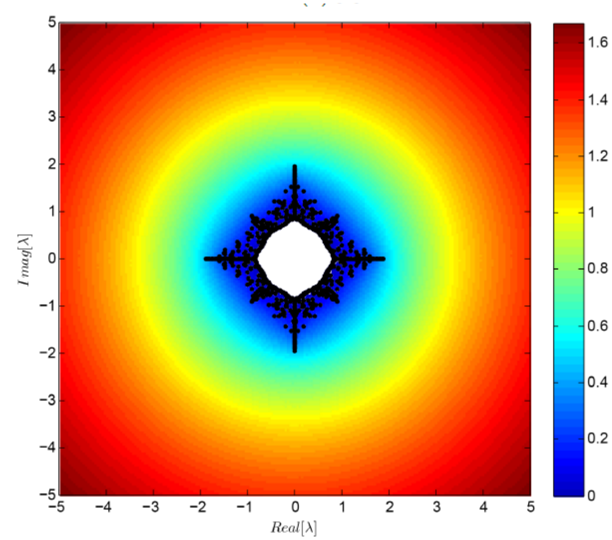




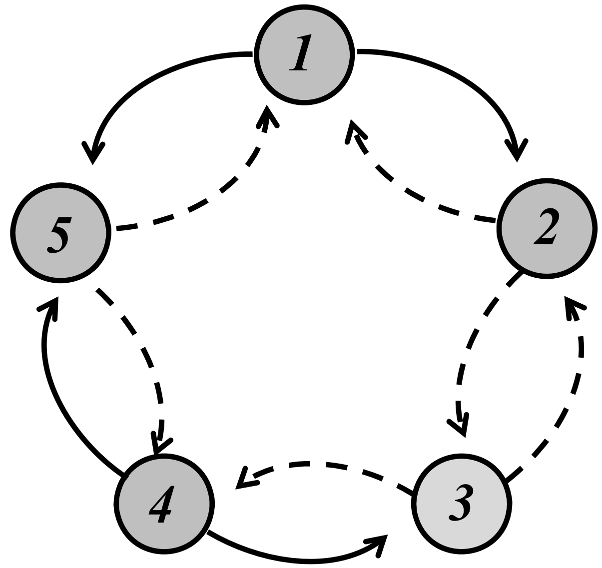
Inverse localization length

for $g > 0$ given by

$$\kappa(E_1, E_2, g) \; = \kappa(E_1, E_2, 0) - g$$



What about Dale's law? All neurons must be purely excitatory or inhibitory….



$N = 5,\ g = 0.0$

$$\boldsymbol{G} = \begin{pmatrix} & + & & & + \\ - & & - & & \\ & - & & - & \\ & & + & & + \\ - & & & - & \end{pmatrix}$$

$$\overleftrightarrow{M} = -\sum_{j=1}^{N}\left[s_j^{+} e^{g} \left| j+1 \right\rangle \left\langle j \right| + s_j^{-} e^{-g} \left| j \right\rangle \left\langle j+1 \right| \right]$$

Replace 2N random variables with only N of them…

$$\overleftrightarrow{G} = -\sum_{k=1}^{N} \sigma_k \left[e^{g} \left| k+1 \right\rangle \left\langle k \right| + e^{-g} \left| k-1 \right\rangle \left\langle k \right| \right]$$

The spectra and eigenfunctionsof M and G are essentially identical! The spectral properties are determined in both cases by above/below diagonal products such as $M(j, j+1)\cdot M(j+1, j) = s_j^{+} s_j^{-}$ and $G(j, j+1)\cdot G(j+1, j) = \sigma_j \sigma_{j+1}$,

which have identical statistics!!

Large g spectra: perturbation theory about a "delay ring"

$$M = \begin{pmatrix} 0 & s_1^- e^g & 0 & \dots & s_N^+ e^{-g} \\ s_1^+ e^{-g} & 0 & s_2^- e^g & 0 & \\ 0 & s_2^+ e^{-g} & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & & s_{N-1}^- e^g \\ s_N^- e^g & \dots & 0 & s_{N-1}^+ e^{-g} & 0 \end{pmatrix}$$

After a similarity transformation $M \to M' = A + B$

$N = 4$

$$A = e^g \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \qquad B = e^{-g} \begin{pmatrix} 0 & 0 & 0 & b_4 \\ b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_3 & 0 \end{pmatrix}.$$

$b_j = s_j^+ s_j^-$

Eigenvectors of A are plane waves

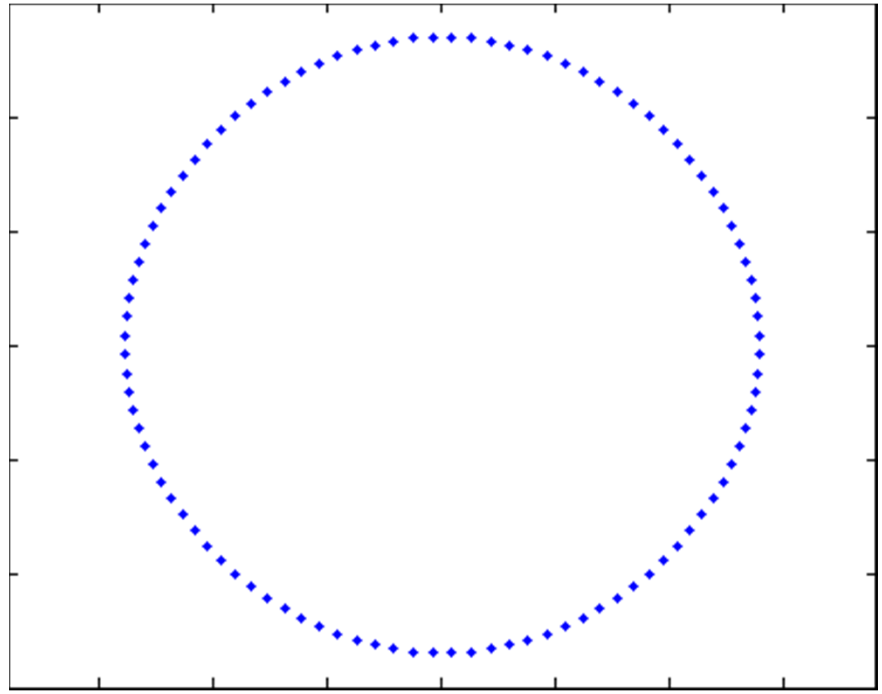
$v_j^R = e^{ikj} \quad v_j^L = e^{-ikj}$

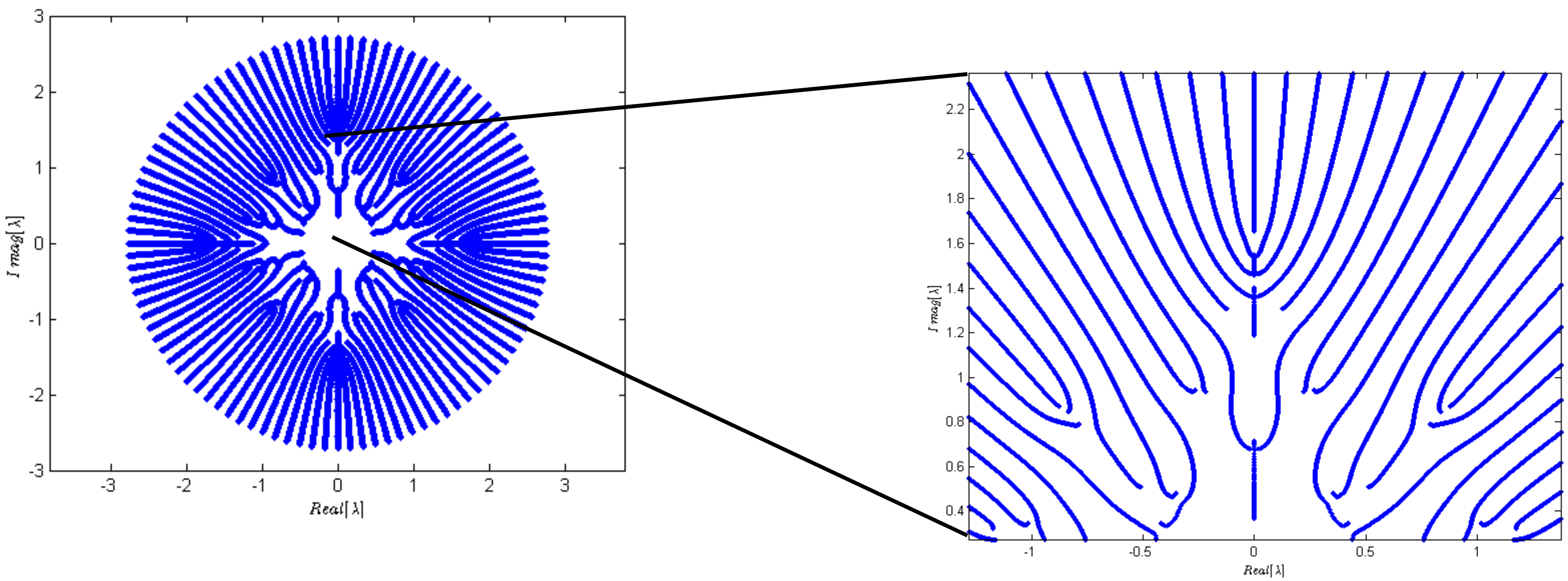
$k = 2\pi n/N, \; n = 0, 1 \dots (N-1)$

$$\lambda_k = e^{g+ik} + e^{-g}[e^{-ik}(b_1 + b_2 \dots + b_{n-1} + b_N)/N] \quad = \quad \text{random walk}$$

Large g limit: Plane wave states, all eigenfunctions delocalized

- *Trajectories of eigenvalues for N=100 and values of g decreasing from 1 down to zero.*
- *Eigenvalues "flow" in the complex plane.*
- *Motion stops once eigenvalues localize*

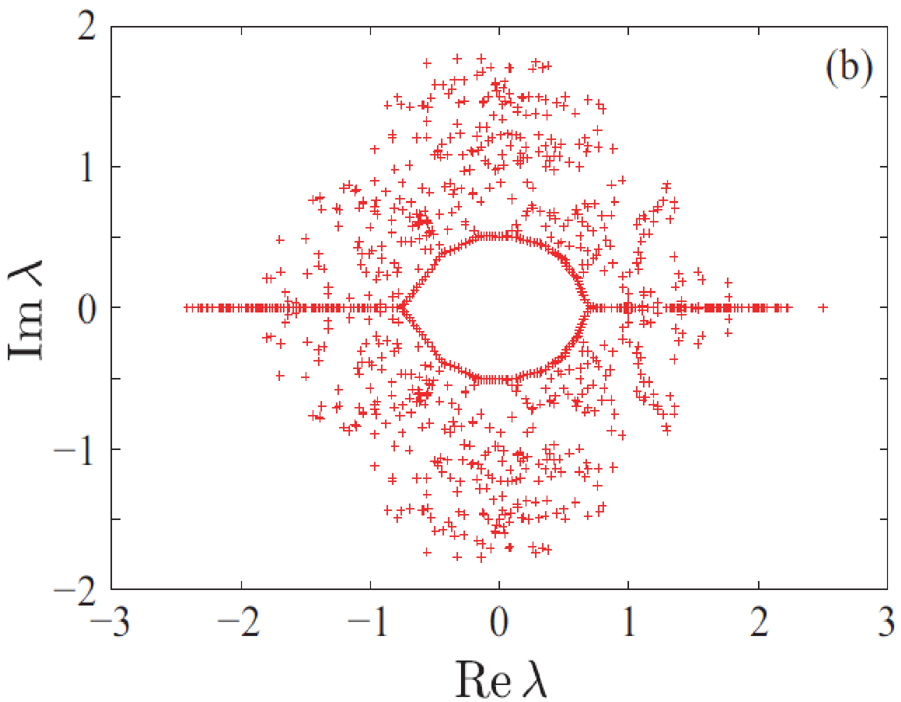




The gap rimmed by extended states is robust…

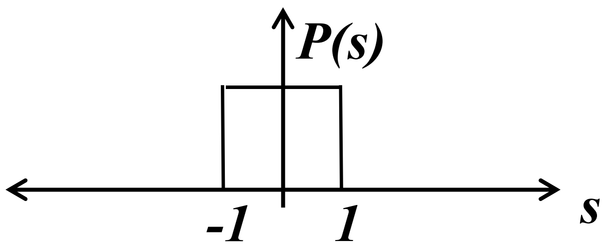
$s_j^+ = \pm 1, \; s_j^- = \pm 1, \; N = 1000,$

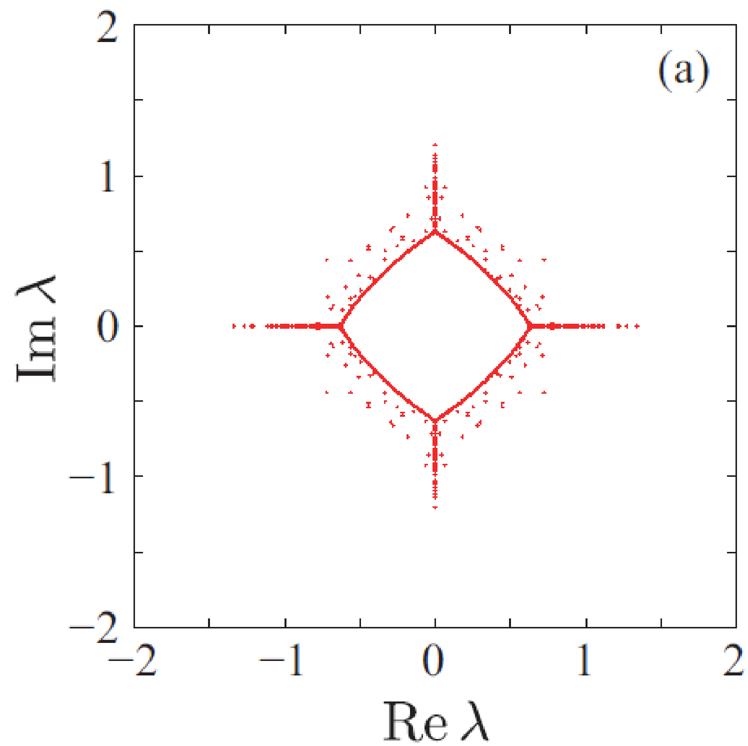
$g =$ 0.1, but with diagonal randomness



Single box distribution

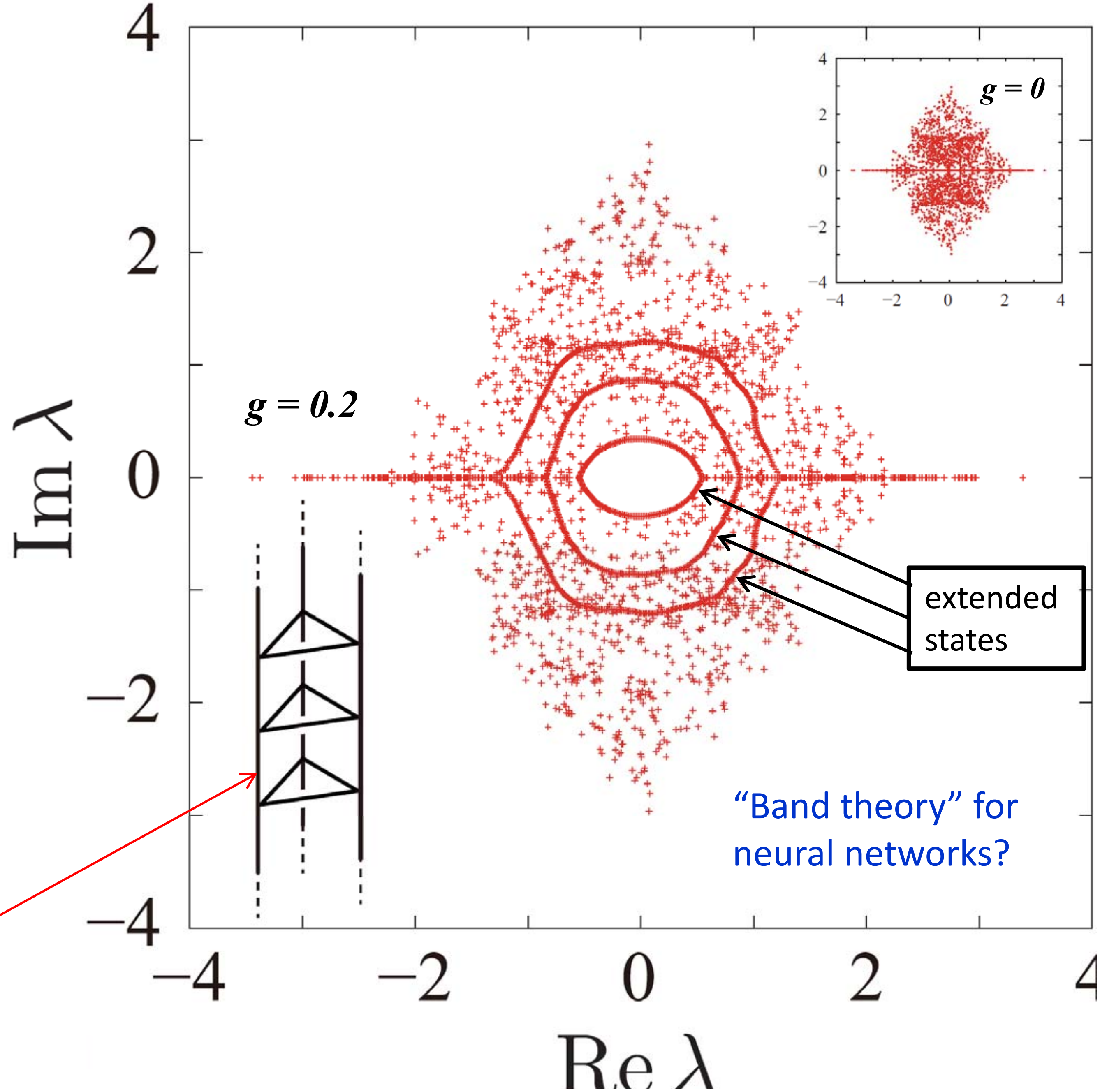
N = 1000, g = 0.5





Energy gap and rings of extended states also appear for coupled neural clusters

1000 triangular neural clusters, obeying Dale's law, and coupled together to form a ring



Layered neural network with tunable back propagation

Non-Hermitian Localization in Ecological and Neural Networks

- *Non-Hermitian matrices, with complex eigenvalue spectra, arise naturally in simple models of complex ecosystems, and neural networks.*

- *Striking departures from the conventional wisdom about localization arise in the one-dimensional non-Hermitian random matrices that describe sparse neural and ecological networks.*

- *An intricate eigenvalue spectrum controls the spontaneous activity and induced response. Directed rings of neurons lead to a hole centered on in the density of states in the complex plane.*

- *All states are extended on the rim of this hole, while the states outside the hole are localized.*

Thank you!

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