Many-body topological invariants for topological superconductors (and insulators)

Shinsei Ryu

The University of Chicago

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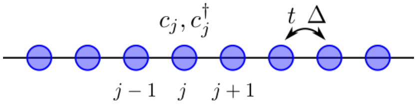
Outline

- Motivations: the Kitaev chain with interactions
- The kitaev chain with reflection symmetry
- The kitaev chain with time-reversal symmetry
- Summary
- Collaborators: Hassan Shapourian (UIUC) and Ken Shiozaki (UIUC)

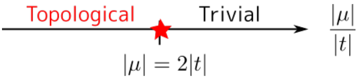
Motivating example: The Kitaev chain

- The Kitaev chain

$$H = \sum_j \Big[-tc_j^\dagger c_{j+1} + \Delta c_{j+1}^\dagger c_j^\dagger + h.c. \Big] - \mu \sum_j c_j^\dagger c_j$$



- Phase diagram: there are only two phases:



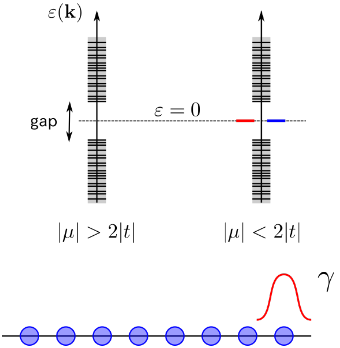
- Topologically non-trivial phase is realized when $2|t| \geq |\mu|$.

- The topological phase is characterized by (i) a bulk $\mathbb{Z}_2$ topological invariant:

$$\exp\left[i\int_{-\pi}^{\pi} dk\, \mathcal{A}_x(k)\right] = \pm 1$$

where $\mathcal{A}(k) = i\langle u(k)|du(k)\rangle$ is the Berry connection.

- (ii) Majorana end states:



- Characterized by the $\mathbb{Z}_2$ topological invariant $\simeq$ the even/odd Majorana end states.

The Kitaev chain with TR or reflection symmetry

- The Kitaev chain can be studied in the presence of time-reversal or reflection symmetry.

$$Tc_jT^{-1} = c_j \; (TiT^{-1} = -i) \quad \text{or} \quad Rc_jR^{-1} = ic_{-j}.$$

- Once we impose more symmetries (T or R), we distinguish more phases. Phases are classified by an integer $\mathbb{Z}$.
- There is a topological invariant written in terms of Bloch wave functions.

$$\nu = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk\, \mathcal{A}_x(k) = (\text{integer})$$

- Ex: For N_f copies of the Kitaev chain $H = \sum_{a=1}^{N_f} H^a$ with $|\mu| < 2|t|$, the topological invariant is $\nu = N_f$.

The Kitaev chain with interactions

- The 1d Kitaev chain: $H = \sum_{a=1}^{N_f} H^a$ with TR or reflection symmetry

$$THT^{-1} = H \quad \text{or} \quad RHR^{-1} = H$$

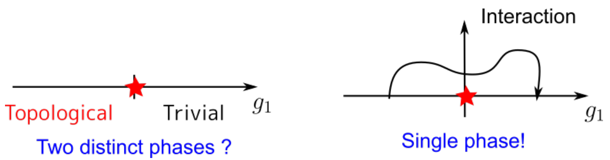
- Add interactions; *Can we destropy the topological phase by interactions?;*

$$H \to H + wV$$

I.e., Is it possible to go from topological to trivial ?

- Interestingly, the answer is Yes! [Fidkowski-Kitaev(10)]
- You can show there is a (rather complicated) interaction V that destroys the topological case.
- *Only possible when* $N_f = 8$ *(*$N_f \equiv 0$ *mod 8)*

Issues and Goal



- Clearly, something is missing in non-interacting classification. We have not explored the phase diagram "hard enough".
- Various other examples in which the non-interacting classification breaks down.
- Non-interacting topological invariants are not enough/spurious.
- Goal: find many-body invariants for fermionic symmetry-protected topological phases.

Main results [arXiv:1607.03896 and arXiv:1609.05970]

- We have succeeded in construction many-body topological invariants for many fermion SPT phases.
- These invariants do not refer to single particle wave functions (Bloch wave functions). They are written in terms of many-body ground states $|\Psi\rangle$.
- C.f. Many-body Chern number.
- Strategy behind the construction (later). [Hsieh-Sule-Cho-SR-Leigh (14), Kapustin et al (14), Witten (15)]

The Kitaev chain with reflection

- Consider *Partial reflection* operation R_{part}, which acts only a part of the system.

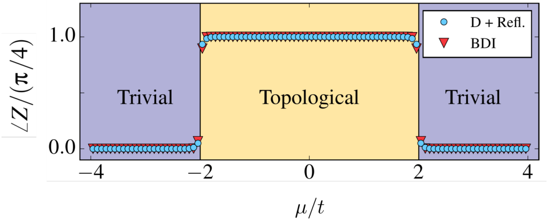
- We claim the phase of the overlap

$$Z = \langle \Psi | R_{part} | \Psi \rangle$$

is quantized, and serves as the many-body topological invariant.

- (Similar but somewhat more complicated invariant for TR symmetric case.)

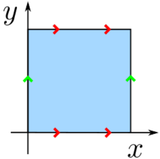
- Numerically check (blue filled circles).

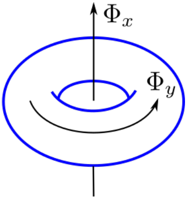


- The phase of Z is the 8th root of unity.

Strategy behind the construction

- In the quantum Hall effect, the many-body Chern number is formulated by putting the system on the spatial torus:





- Introduce the twisting boundary condition by $U(1)$, and measure the response: Ground state on a torus with flux $|\Psi(\Phi_x, \Phi_y)\rangle$
- Berry connection in parameter space

$$A_i = i\langle\Psi(\Phi_x, \Phi_y)|\frac{\partial}{\partial\Phi_i}|\Psi(\Phi_x, \Phi_y)\rangle$$

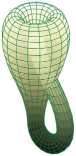
- *Many-body Chern number* [Niu-Thouless-Wu (85)]:

$$Ch = \frac{1}{2\pi}\int_0^{2\pi} d\Phi_x \int_0^{2\pi} d\Phi_y (\partial_{\Phi_x} A_y - \partial_{\Phi_y} A_x)$$

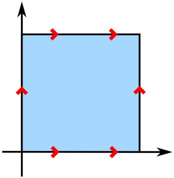
Strategy behind the construction

- A famous saying

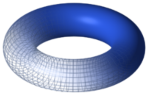
- However, new phases of matter requires new kinds of manifolds, *unoriented* manifolds. E.g. Klein bottle

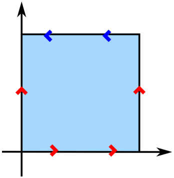


- For topological phases with reflection symmetry, we twist boundary conditions using the symmetry of the problem (reflection)

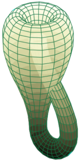


$$\Psi(x+L,y) = e^{i\Phi_x}\Psi(x,y)$$
$$\Psi(x,y+L) = e^{i\Phi_y}\Psi(x,y)$$

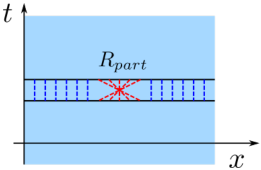


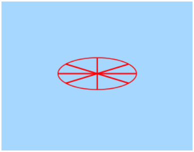


$$\Psi(t+T,x) = \Psi(t,-x)$$



- Partial reflection introduces a "crosscap" in space time:





- The spacetime is effectively *the real projective plane*, $\mathbb{R}P^2$.

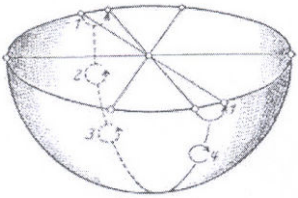


Abb. 9.

More general considerations

- For condensed matter systems with symmetries, we can consider

$$Z(X, \eta, A) = \int \mathcal{D}[\phi_i] \exp -S(X, \eta, A, \phi_i) \qquad (1)$$

X: spacetime, η: "structure" (e.g., spin, pin, etc.), A: background gauge field, ϕ_i: matter field.

- For orientation-reversing symmetries (time-reversal, reflection, etc.), unoriented spacetime (with pin structure) plays the role of an external background field.
- For gapped topological phases, we expect $Z(X, \eta, A)$ consists of a topological term:

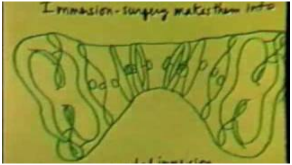
$$Z(X, \eta, A) \sim \exp[iS_{top}(X, \eta, A, \cdots) + \cdots] \qquad (2)$$

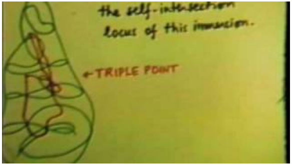
- S_{top} are our many-body topological invariants.
- For SPT phases (i.e., no topological order, unique ground state), S_{top} are expected to be classified by cobordism theory. [Kapustin et al (14), Freed et al(14-16)]

Why does the real projective plane know "8"?

- Interesting mathematics... [Kapustin et al (14), Witten (15), Freed et al(14-16)]
- Watch a Youtube video (thanks: Dennis Sullivan):

  https://www.youtube.com/watch?v=7ZbbhBQEJmI





Topology: The connected sum of 8 copies of Boys Surface is immersion-cobordant to zero. The addition and the cobordism are illustrated here.

Higher dimensions – (3+1)d with inversion

- Consider ^{3}He-B:

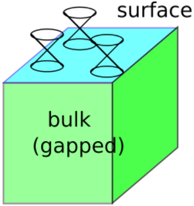
$$H = \int d^3\mathsf{k}\, \Psi^\dagger(\mathsf{k})\mathcal{H}(\mathsf{k})\Psi(\mathsf{k}), \quad \mathcal{H}(\mathsf{k}) = \begin{bmatrix} \frac{k^2}{2m} - \mu & \Delta\sigma\cdot\mathsf{k} \\ \Delta\sigma\cdot\mathsf{k} & -\frac{k^2}{2m} + \mu \end{bmatrix}$$

$$\Psi(\mathsf{k}) = (\psi_\uparrow(\mathsf{k}), \psi_\downarrow(\mathsf{k}), \psi_\downarrow^\dagger(-\mathsf{k}), -\psi_\uparrow^\dagger(-\mathsf{k}))^T$$

- Inversion symmetry:

$$I\psi_\sigma^\dagger(\mathsf{r})I^{-1} = i\psi_\sigma^\dagger(-\mathsf{r})$$

- Topologically protected surface Majorana cone (stable when the surface is inversion symmetric)

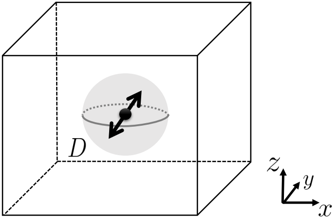


- Characterized by the integer topological invariant at non-interacting level.

Many-body topological invariant

- Previous studies indicate the non-interacting classification breaks down to $\mathbb{Z}_{16}$. Surface topological order. [Fidkowski et al (13), Metlitski et al (14), Wang-Senthil (14)]
- We consider partial inversion I_{part} on a ball D:

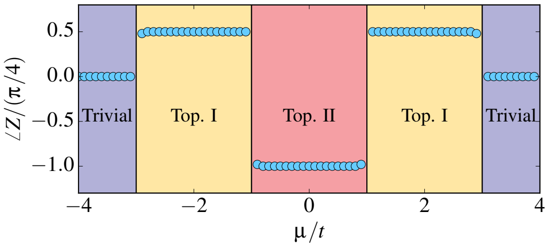
$$Z = \langle \Psi | I_{part} | \Psi \rangle$$



- The spacetime is effectively four-dimensional projective plane, $\mathbb{R}P^4$.

Calculations

- Numerics on a lattice:



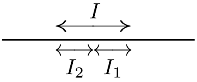
- Matches with the analytical result

$$Z = \exp\left[-\frac{i\pi}{8} + \frac{1}{12}\ln(2) - \frac{21}{16}\zeta(3)\left(\frac{R}{\xi}\right)^2 + \cdots\right]$$

- C.f. topologically ordered surfaces: [Wang-Levin, Tachikawa-Yonekura, Barkeshli et al (16)]

The Kitaev chain with Time-reversal – “partial time-reversal”

- Start from the reduced density matrix for the interval I, $\rho_I := \mathrm{Tr}_{\bar{I}}|\Psi\rangle\langle\Psi|$.
- I consists of two *adjacent* intervals, $I = I_1 \cup I_2$.

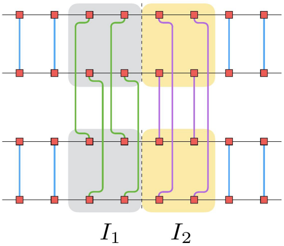


- We consider *partial time-reversal* acting only for I_1; $\rho_I \longrightarrow \rho_I^{T_1}$.
- Partial time reversal $\simeq$ partial transpose has to be properly defined for fermionic systems [Shaporian-Shiozaki-Ryu 17];
- (led to new entanglement measure, fermionic entanglement negativity.)
- The invariant:

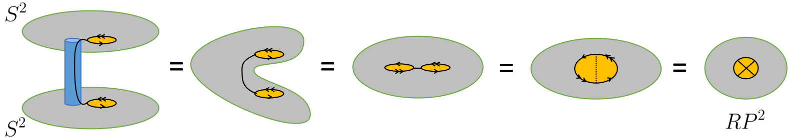
$$Z = \mathrm{Tr}[\rho_I \rho_I^{T_1}],$$

▶

$$Z = \mathrm{Tr}[\rho_I \rho_I^{T_1}],$$



- The invariant "simulates" the path integral on $\mathbb{R}P^2$:

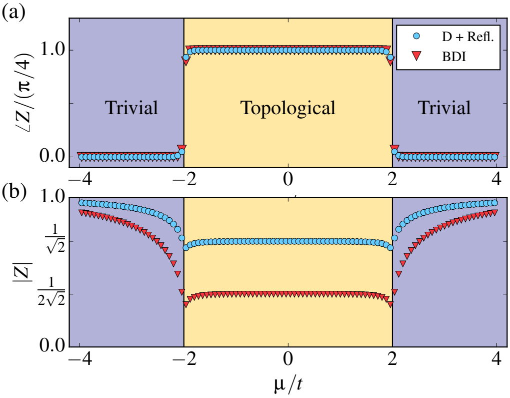


Calculations

- Analytical calculations in the zero correlation length limit:

$$\mathrm{Tr}(\rho_I \rho_I^{T_1}) = \frac{1+i}{4} = \frac{1}{2\sqrt{2}} e^{i\pi/4}$$

- Numerically checked away from the zero correlation length (red triangles).



More on partial transpose

- The partial transpose for bosons; definition: for the density matrix $\rho_{A_1 \cup A_2}$,

$$\langle e_i^{(1)} e_j^{(2)} | \rho_{A_1 \cup A_2}^{T_2} | e_k^{(1)} e_l^{(2)} \rangle = \langle e_i^{(1)} e_l^{(2)} | \rho_{A_1 \cup A_2} | e_k^{(1)} e_j^{(2)} \rangle$$

  where $|e_i^{(1,2)}\rangle$ is the basis of $\mathcal{H}_{A_1, A_2}$.

- Can detect quantum correlation comes from off diagonal parts of density matrices. [Peres (96), Horodecki-Horodecki-Horodecki (96), Vidal-Werner (02), Plenio (05) ...]

- Entanglement negativity and logarithmic negativity

$$\frac{1}{2}(\mathrm{Tr}\,|\rho_A^{T_2}| - 1), \quad \mathcal{E}_A = \log\, \mathrm{Tr}\,|\rho_A^{T_2}|$$

Issues in fermionic systems

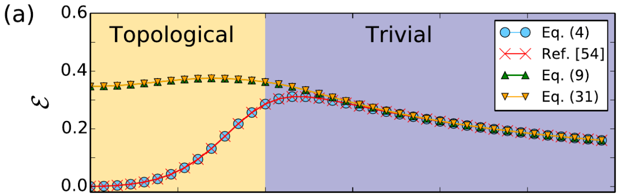
- Fermionic version of the partial tranpose has been considered previously. E.g., [Eisler-Zimborás]
- Two issues:
  - (i) Partial transpose of fermionic Gaussian states are not Gaussian
    - Partial transpose of bosonic Gaussian states is still Gaussian; easy to compute by using the correlation matrix
    - ρ^{T_1} can be written in terms of two Gaussian operators $O_\pm$:

$$\rho^{T_1} = \frac{1-i}{2} O_+ + \frac{1+i}{2} O_-$$

    - Negativity estimators/bounds using $\mathrm{Tr}\,[\sqrt{O_+ O_-}]$ [Herzog-Y. Wang (16), Eisert-Eisler-Zimborás (16)]
    - Spin structures: [Coser-Tonni-Calabrese, Herzog-Wang]
  - (ii) Fails to capture "Majorana dimers" in the Kitaev chain
- Our new definition of partial time-reversal solves these issues

Issues in fermionic systems (2): The Kitaev chain

- Consider log negativity $\mathcal{E}$ for two adjacent intervals of equal length. $(L = 4\ell = 8)$



- Vertical axis: μ/t ranging from 0 to 6.

- (Blue circles) is computed for the bosonic many-particle density matrix in the Ising chain with periodic boundary condition.

- (Red crosses) curves are computed for the fermionic many-particle density matrix according to the rules by Eisler-Zimborás.

- Log negativity fails to capture Majorana dimers.

Partial transpose for fermions – our definition

- In the Majorana representation,

$$\begin{aligned}\rho_A^{R_1} &= \sum_{\kappa,\tau} w_{\kappa,\tau}\, \mathcal{R}(c_{m_1}^{\kappa_1}\cdots c_{m_{2k}}^{\kappa_{2k}})\, c_{n_1}^{\tau_1}\cdots c_{n_{2l}}^{\tau_{2l}} \\ &= \sum_{\kappa,\tau} w_{\kappa,\tau}\, i^{|\kappa|}\, c_{m_1}^{\kappa_1}\cdots c_{m_{2k}}^{\kappa_{2k}}\, c_{n_1}^{\tau_1}\cdots c_{n_{2l}}^{\tau_{2l}}\end{aligned}$$

  where $\mathcal{R}$ satisfies: $\mathcal{R}(c) = ic$, $\mathcal{R}(M_1 M_2) = \mathcal{R}(M_1)\mathcal{R}(M_2)$

- Furthermore,

$$(\rho_A^{R_1})^{R_2} = \rho_A^R, \quad (\rho_A^R)^R = \rho_A, \quad (\rho_A^1 \otimes \cdots \otimes \rho_A^n)^{R_1} = (\rho_A^1)^{R_1} \otimes \cdots \otimes (\rho_A^n)^{R_1}$$

- Fermions Gaussian states stay Gaussian after partial TR,

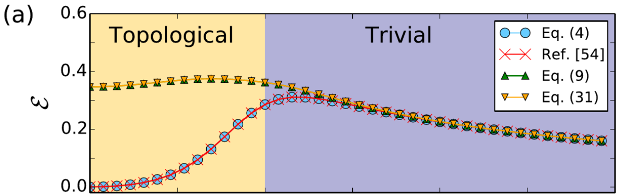
$$\rho_A^{R_1} = O_+, \quad \rho_A^{R_1\,\dagger} = O_-,$$

  Possible to develop the free fermion formula.

- *Define* log negativity as

$$\mathcal{E} := \ln \mathrm{Tr}\, |\rho_A^{R_1}| = \ln \mathrm{Tr}\, \sqrt{O_+ O_-}$$

- Comparison:



  - (Green triangles) are computed for the fermionic many-particle density matrix according to our rule. (Computed numerically)
  - (Orange triangles) is computed using the free fermion formula:

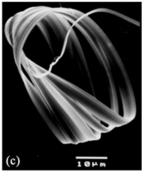
$$\mathcal{E} = \ln\left[\sum_{i=1}^{M}(\sqrt{\lambda_i} + \sqrt{1-\lambda_i})\sqrt{\frac{\mathrm{Pf}\,(\tilde{S} - i\sigma_2)}{\mathrm{Pf}\,(S - i\sigma_2)^2}}\right]$$

Summary

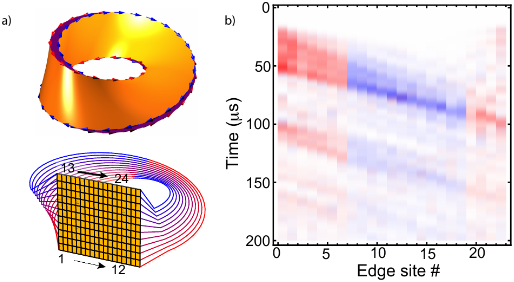
- We have succeeded in constructing many-body topological invariants for SPT phases.
- These invariants do not refer to single particle wave functions (Bloch wave functions). They are written in terms of many-body ground states $|\Psi\rangle$.
- Analogous to go from the single-particle TKNN formula to to the many-body Chern number.
- Many-body invariants in other cases (e.g., time-reversal symmetric topological insulators) can be constructed in a similar way.

Outlook

- Many future applications, in particular, in numerics.
- And ...?



$NbSe_3$ Möbius strip



[Ningyuan-Owens-Sommer-Schuster-Simon (13)]