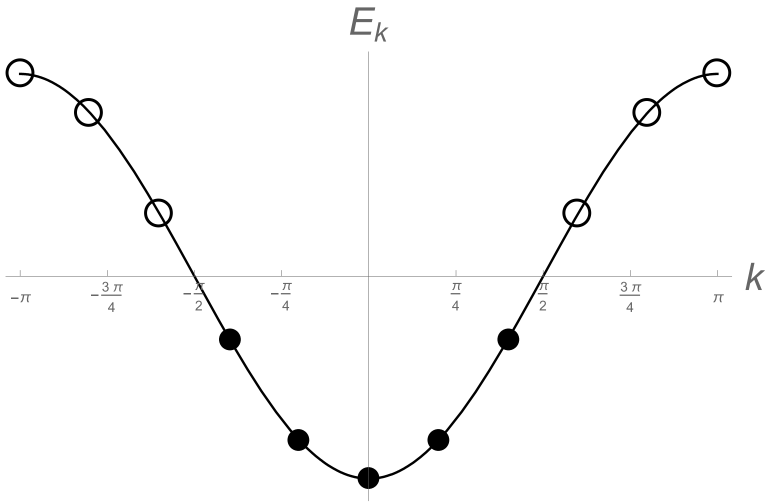
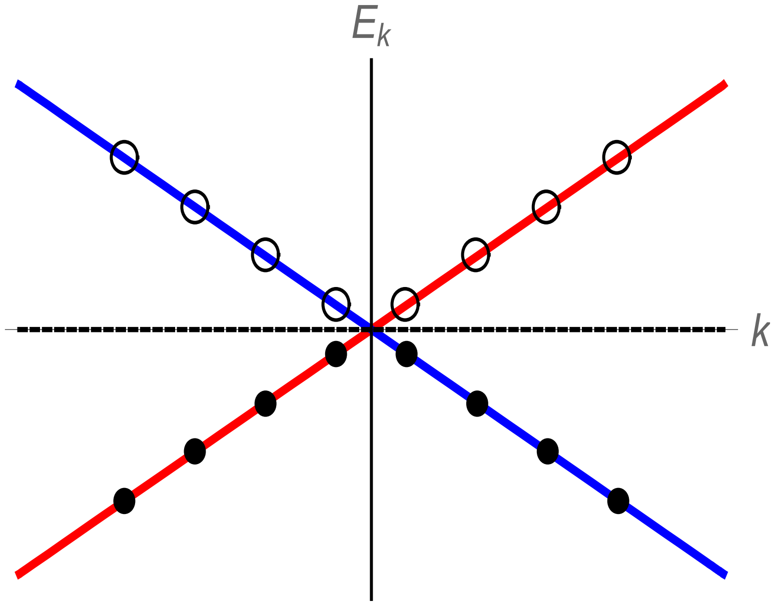
Lieb-Schultz-Mattis Theorems for Symmetry Protected Topological phases



Yuan-Ming Lu



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Outline

❑ Motivation

- Introduction to *non-interacting* TIs and *interacting* SPTs
- Introduction to the Lieb-Schultz-Mattis theorem and its descendants

❑ Results

- Magnetic translation symmetry and setup of the problem
- Summary of results for interacting fermions and/or bosons
- Applications to physical systems

❑ Sketch of proofs/arguments

- Chiral TSC in class D: Fermion parity switch (*arXiv* ***1705.04691***)
- IQHE of bosons/fermions: Polarization (*arXiv* ***1705.09298***)
- QSHE and SPTs: Entanglement spectrum (*arXiv* ***1703.04776***)

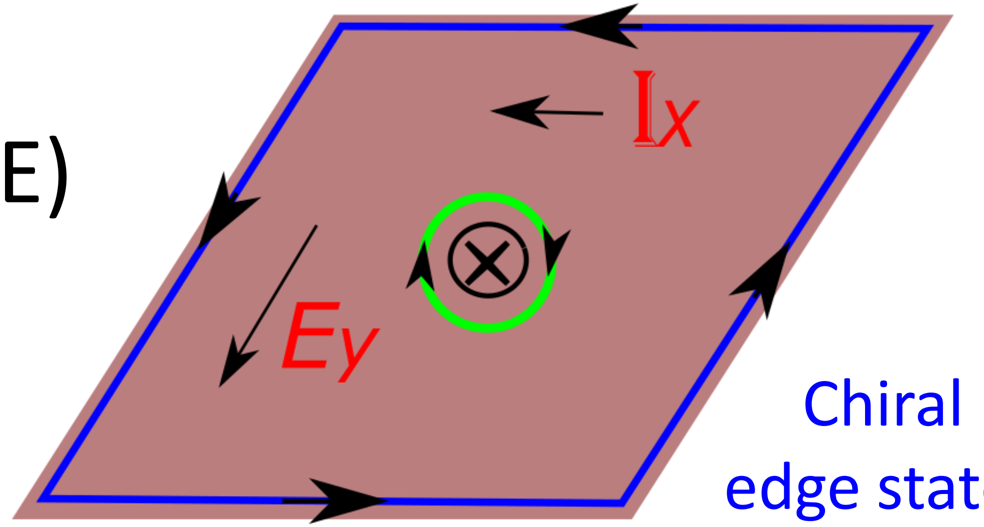
Concepts of Topological Insulators (TIs)

Example 1:
Integer quantum Hall effect (IQHE)

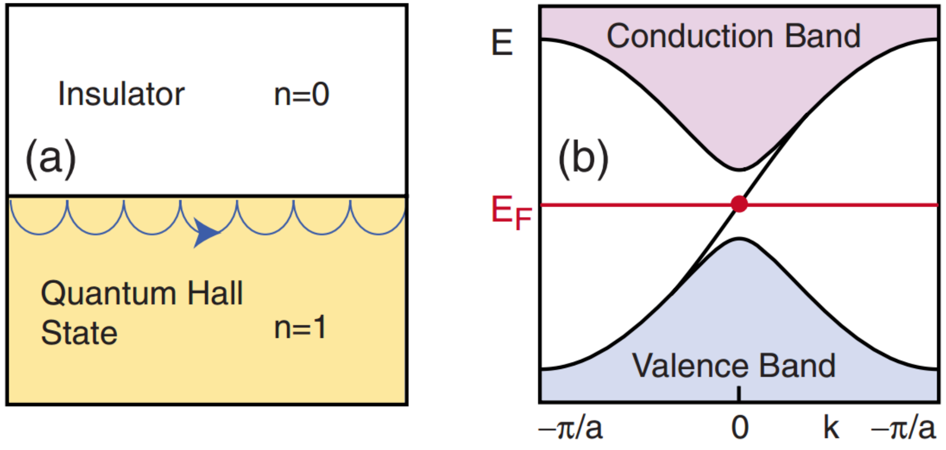
- Insulating (localized) bulk states

$$\sigma_{xy} = n\frac{e^2}{h}, \quad n \in \mathbb{Z}$$

- Conducting (delocalized) edge/surface states



Chiral edge states



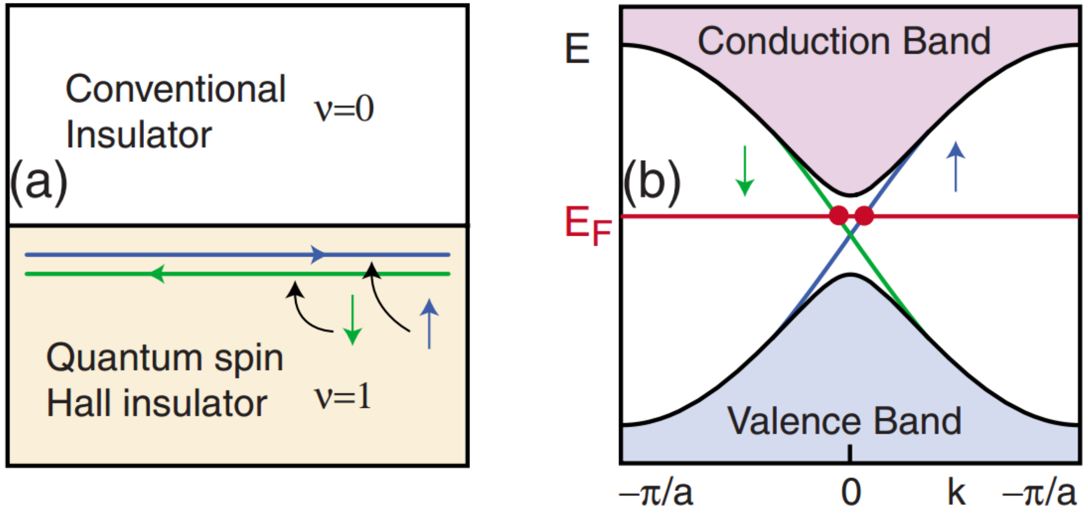
Bulk-boundary correspondence

Fig. from *Hasan, Kane (2010)*

Example 2: Quantum spin Hall effect (QSHE)

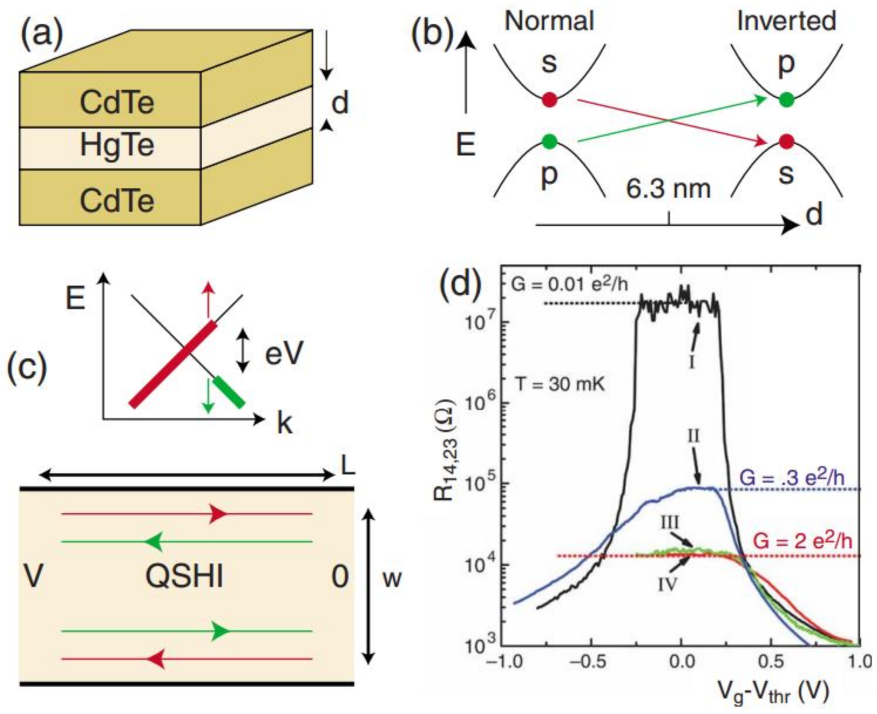
- Time reversal symmetry → *Z2*-valued topological index (*Kane, Mele 2005, Bernevig, Hughes, Zhang 2005,…*)

$$\nu = 0, 1 \in \mathbb{Z}_2$$



Helical edge states:
protected by **time reversal symmetry**

Fig. from *Hasan, Kane (2010)*



Konig et al (2007)

Classification of *non-interacting* fermionic TIs

- Mathematical setup

$$\hat{H} = \sum_{a,b} f_a^\dagger H_{a,b} f_b$$

$$\{f_a, f_b\} = 0$$

Grassmann #'s

$$\{f_a, f_b^\dagger\} = \delta_{a,b}$$

$$H = H^\dagger$$

(i) H is *quasi-diagonal* (***locality***)

(ii) H is ***gapped***

class	$d=0$	1	2	3
A	$\mathbb{Z}$		$\mathbb{Z}$	
AIII		$\mathbb{Z}$		$\mathbb{Z}$
AI	$\mathbb{Z}$			
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$		
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
DIII		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII		$2\mathbb{Z}$		$\mathbb{Z}_2$
C			$2\mathbb{Z}$	
CI				$2\mathbb{Z}$

Physical Symmetry

A → $U(1)_{\text{charge}}$

D → No symmetry

DIII → Time reversal $\mathcal{T}$

AII → $U(1)_{\text{charge}} \rtimes \mathcal{T}$

Schnyder, Ryu, Furusaki, Ludwig (2008) *Kitaev (2009)*

Generalization to interacting particles?

- Interacting fermions?

$$\hat{H}_{\rm int} = \sum_{a,b} f_a^\dagger H_{a,b} f_b + \sum_{a,b,c,d} U_{ab,cd} f_a^\dagger f_b^\dagger f_c f_d + \cdots$$

- Interacting bosons/spins?

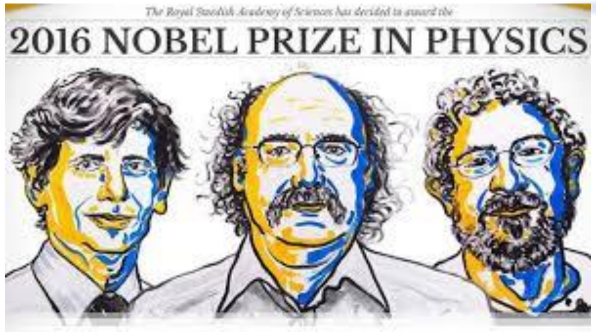
(Typically they form *Bose-Einstein condensate/Magnetic orders* if non-interacting)

- Classification of interacting topological states w/ symmetry protected surface/edge states?

Symmetry Protected Topological (SPT) states

- ***Gapped*** local Hamiltonian
- ***Unique*** ground state on any closed manifold (*No g.s. degeneracy*)
- Symmetry protected ***surface/edge states*** on an open boundary

Simplest example: *d=1 Haldane phase in AKLT model*



Haldane (1983), Affleck, Kennedy, Lieb, Tasaki (1987)

$$\mathcal{H}^{1+1}\left[Z_2^T, U(1)\right] = \mathbb{Z}_2$$

Where to find *interacting* SPT states in *d>1*?

- Higher than 1 dimension: hard to realize SPTs (*other than unrealistic but exactly solvable models*)

- How to **solve the g.s. wavefunction of an interacting system**?

Interacting $2^N\text{-dim. vs. } N\text{-dim.}$ *Non-interacting*

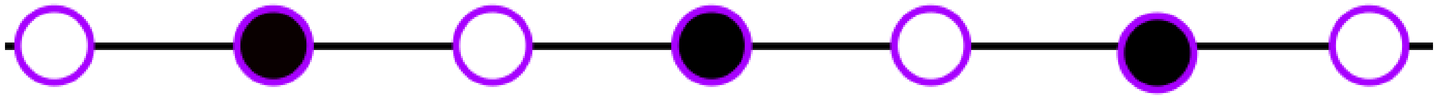
- How to **compute topological invariant of SPTs** from the many-body ground state?

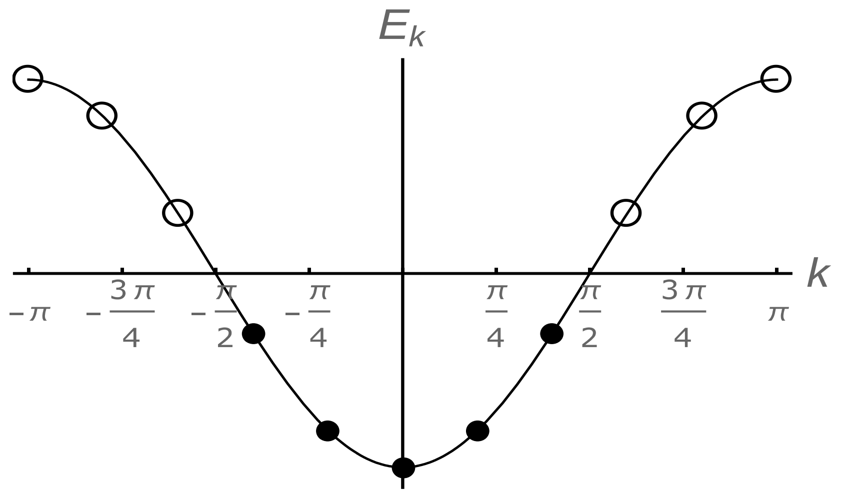
Any magic trick
without computing the g.s.? → **Lieb-Schultz-Mattis Theorems for interacting systems**

The Lieb-Schultz-Mattis (LSM) theorem

- Simplest example: metals vs. insulators?

Half-filling: ½ electron per unit cell
Lattice translation symmetry
U(1) charge conservation ***symmetry***





Impossible to have *an insulator preserving translation sym.!*

No-go theorem, applicable to ***interacting bosons/fermions***

Lieb, Schultz, Mattis (1961) *Oshikawa (2000) Hastings (2004) Parameswaran et al (2013) Roy (2012) Watanabe et al (2015) Cheng et al (2016) Po et al (2017) Jian et al (2017)…*

Ingredients of a LSM Theorem

Microscopic (*Ultraviolet*) input:

Hilbert space (*e.g. filling*) + Translation Sym. + Global Sym.

Microscopic (*Infrared*) Output: Ground State Properties

A unique gapped ground state

Powerful: apply to interacting bosons and/or fermions in different spatial dimensions

Main results: a new class of LSM-type theorem for SPT phases in $d=2$

Microscopic (*Ultraviolet*) input:

Hilbert space + ***Magnetic Translation Sym.*** + Global Sym.

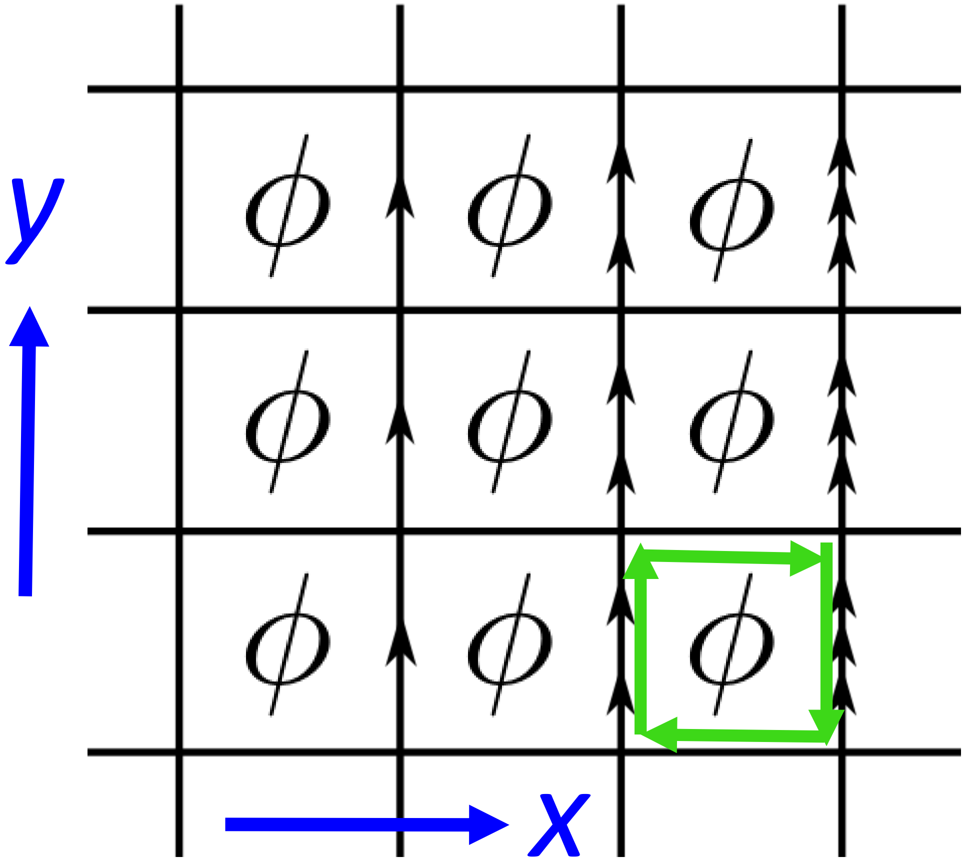
Microscopic (*Infrared*) Output: Ground State Properties

A unique gapped ground state must be a SPT (invertible) phase!

Magnetic Translation Symmetry

- Uniform magnetic field →

Magnetic translation symmetry = *Translation* + *large gauge transformation*



$$\tilde{T}_x = T_x \cdot e^{\mathrm{i}\,\phi \sum_{\mathbf{r}} y \hat{n}_{\mathbf{r}}}$$

$$\tilde{T}_y = T_y$$

Magnetic translation algebra:

$$\tilde{T}_x \tilde{T}_y \tilde{T}_x^{-1} \tilde{T}_y^{-1} = e^{\mathrm{i}\,\phi \sum_{\mathbf{r}} \hat{n}_{\mathbf{r}}}$$

Zak (1964)

LSM theorem for fermionic SPT/invertible phases in *d=2*

Topological superconductors

Physical systems		Microscopic input			Output of LSM theorem		
AZ[25] class	Applications	Symmetry group	d.o.f. per unit cell	Flux per unit cell	Topological invariant	Edge states	Chiral central charge
D	Majorana vortex lattice[26]	$(-1)^{\hat{F}}$	Odd Majoranas $\{\gamma_1, \cdots, \gamma_{2N-1}\}$	$\phi = \pi$	$\nu =$ odd $\in \mathbb{Z}$	Chiral Majorana	$c_- = \nu/2$
DIII	Kitaev-type spin liquid[27]	$\hat{\mathcal{T}}^2 = (-1)^{\hat{F}}$	Odd Majorana Kramers pairs	$\phi = \pi$	$\nu = 1$ $\in \mathbb{Z}_2 = \{0, 1\}$	Helical Majorana	$c_- = 0$
A	Integer QHE in Hofstadter models	$U(1)_{\text{charge}}$	Charge $e \cdot \bar{\rho}_f$	$\phi = 2\pi \frac{p}{q}$	$p\sigma_{xy} = q\bar{\rho}_f \mod q$ $\sigma_{xy} \in \mathbb{Z}$	Chiral fermion	$c_- = \sigma_{xy}$ mod 8[20]
AII	QSHE in π-flux model[28]	$U(1)_{\text{charge}}$ $\hat{\mathcal{T}}^2 = (-1)^{\hat{F}}$	Charge $e \cdot \bar{\rho}_f$ $(\bar{\rho}_f = \text{odd})$	$\phi = \pi$	$\nu = 1$ $\in \mathbb{Z}_2 = \{0, 1\}$	Helical fermion	$c_- = 0$

Topological insulators

*"LSM theorem for SPT phases", YML, arXiv **1705.04691***

Application 1: **IQHE** (***class A***) in Hofstadter models

$$\sigma_{xy} \cdot \frac{\phi}{2\pi} = \bar{\rho} \mod 1$$

Hall conductance in unit of e^2/h

Flux per unit cell

Charge (filling) per unit cell

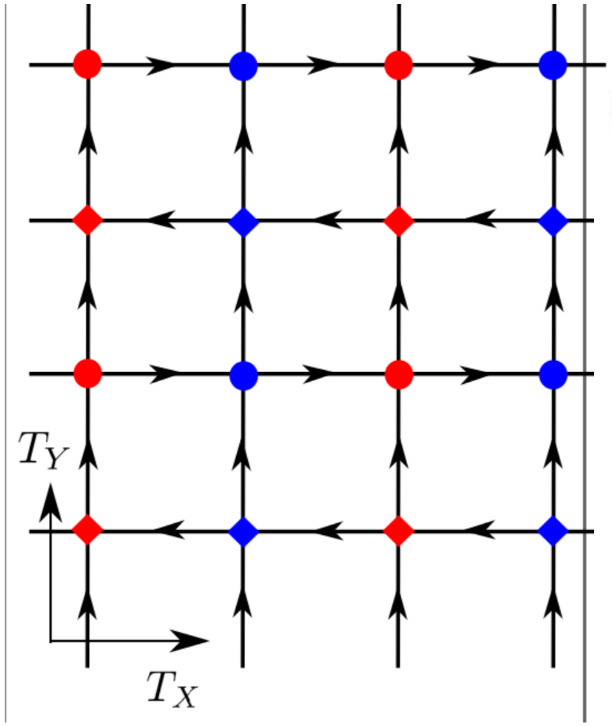
Half-filling: $\bar{\rho} = \frac{1}{2}$; π-flux: $\phi = \pi$ ==> $\sigma_{xy} = 1 \mod 2$

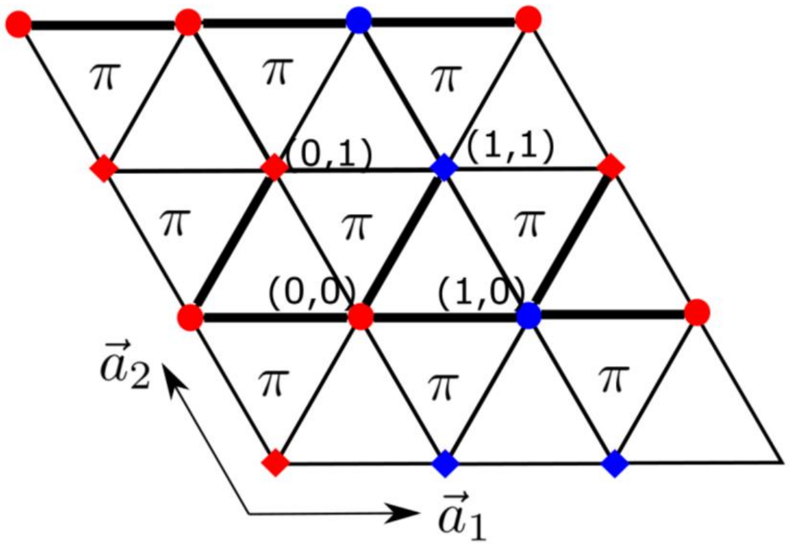
Non-interacting proof: Dana, Avron, Seiler (1985)

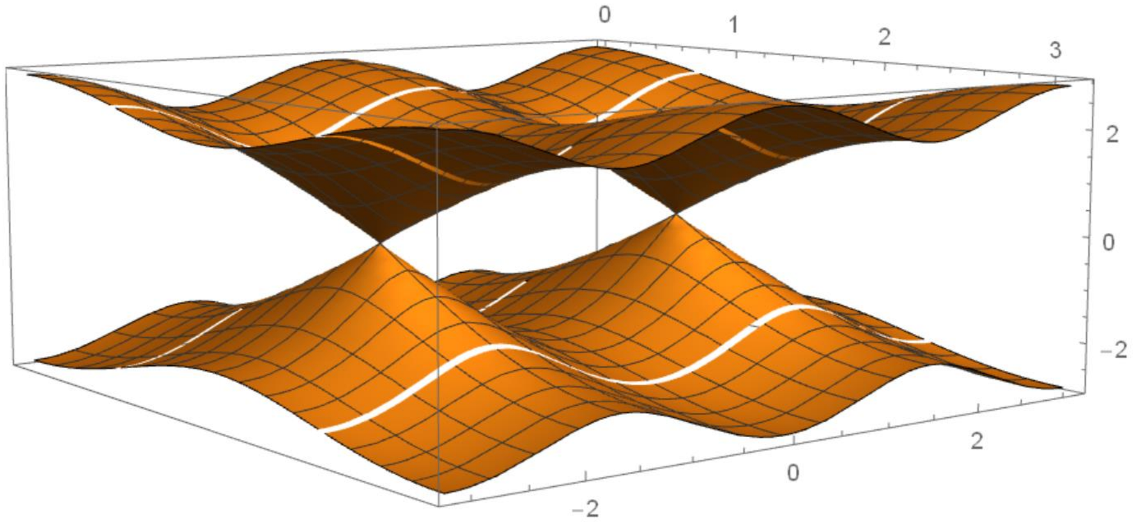
Interacting proof: YML, Ran, Oshikawa, arXiv ***1705.09298***

Example 1.1: half-filling + π-flux Hofstadter models

- π-flux per unit cell at half filling (half electrons per site)
- Square/triangular/kagome lattices……







Common feature:

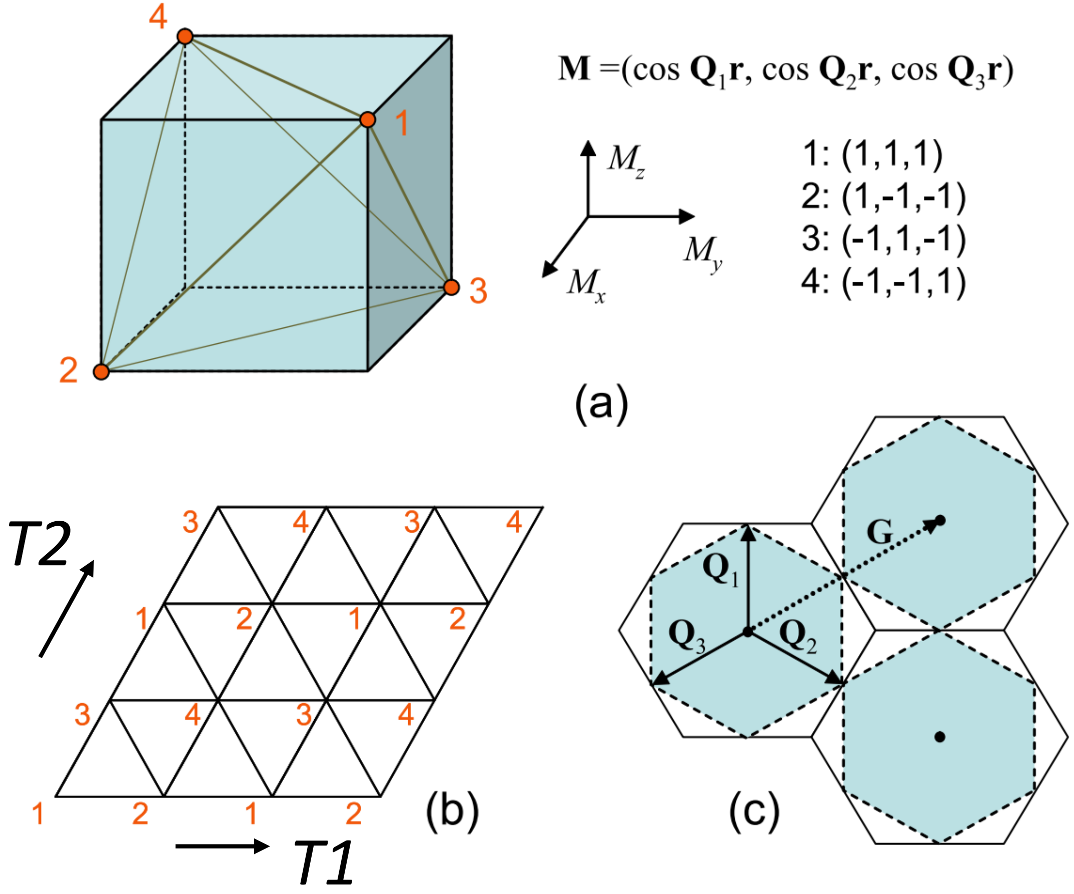
- Two Dirac cones with N.N. hoppings
- Only mass term preserving magnetic translation →

IQHE with Chern number ±1

Example 1.2: magnetic insulators

- “Quarter doping” on triangular/honeycomb/kagome lattices:

  Van-Hove singularity and perfect nesting → interaction-driven “chiral SDW”



$$\tilde{T}_1 = T_1 e^{\mathrm{i}\pi S^x}, \quad \tilde{T}_2 = T_2 e^{\mathrm{i}\pi S^y}$$

$$\tilde{T}_1 \tilde{T}_2 = -\tilde{T}_2 \tilde{T}_1 \Longrightarrow \phi = \pi$$

Quarter filling for spin-1/2 electrons: $\bar{\rho} = \frac{1}{2}$

$$\sigma_{xy} = 1 \mod 2$$

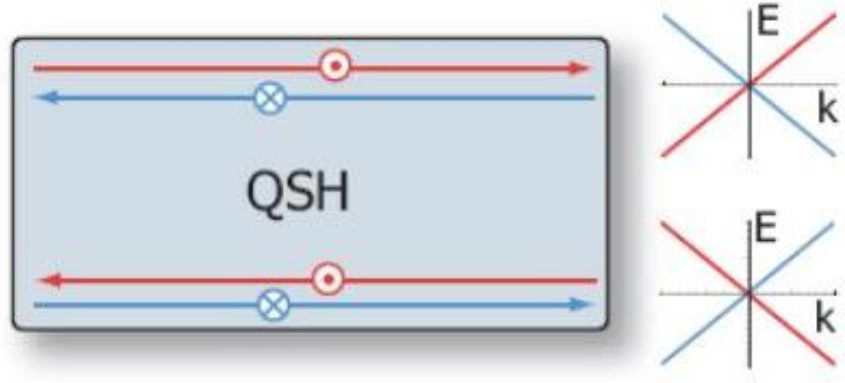
Martin, Batista, PRL (2008) Tao Li, EPL (2012)

Application 2: **QSHE** (***class AII***) in π-flux models

- **Input:** $\bar{\rho}_f \in \mathbb{Z}$ fermions **+** $\phi = \pi$ flux per u.c.
  U(1) charge conservation + ***Time reversal symmetry***

- **Output:** Topological index $\nu = \bar{\rho}_f \mod 2$ for any unique gapped ground state

- *A unique gapped g.s.* ***must exhibit QSHE with odd fermions per unit cell !***



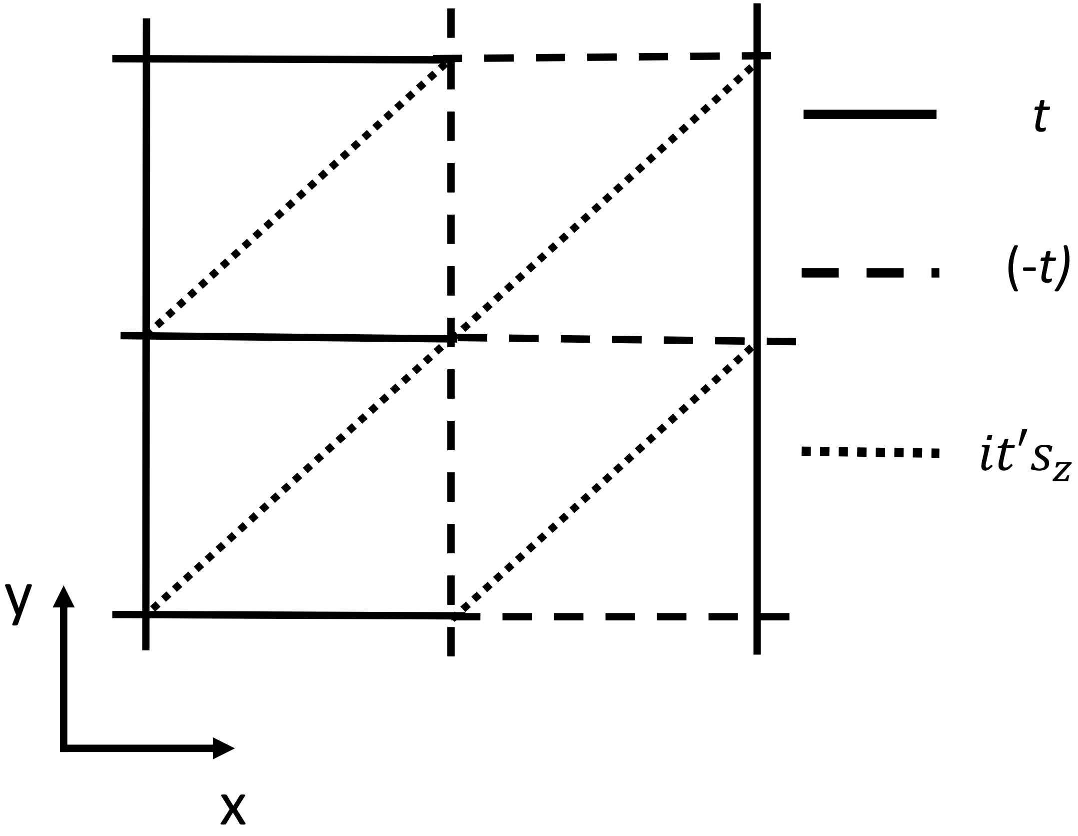
massless Dirac fermions

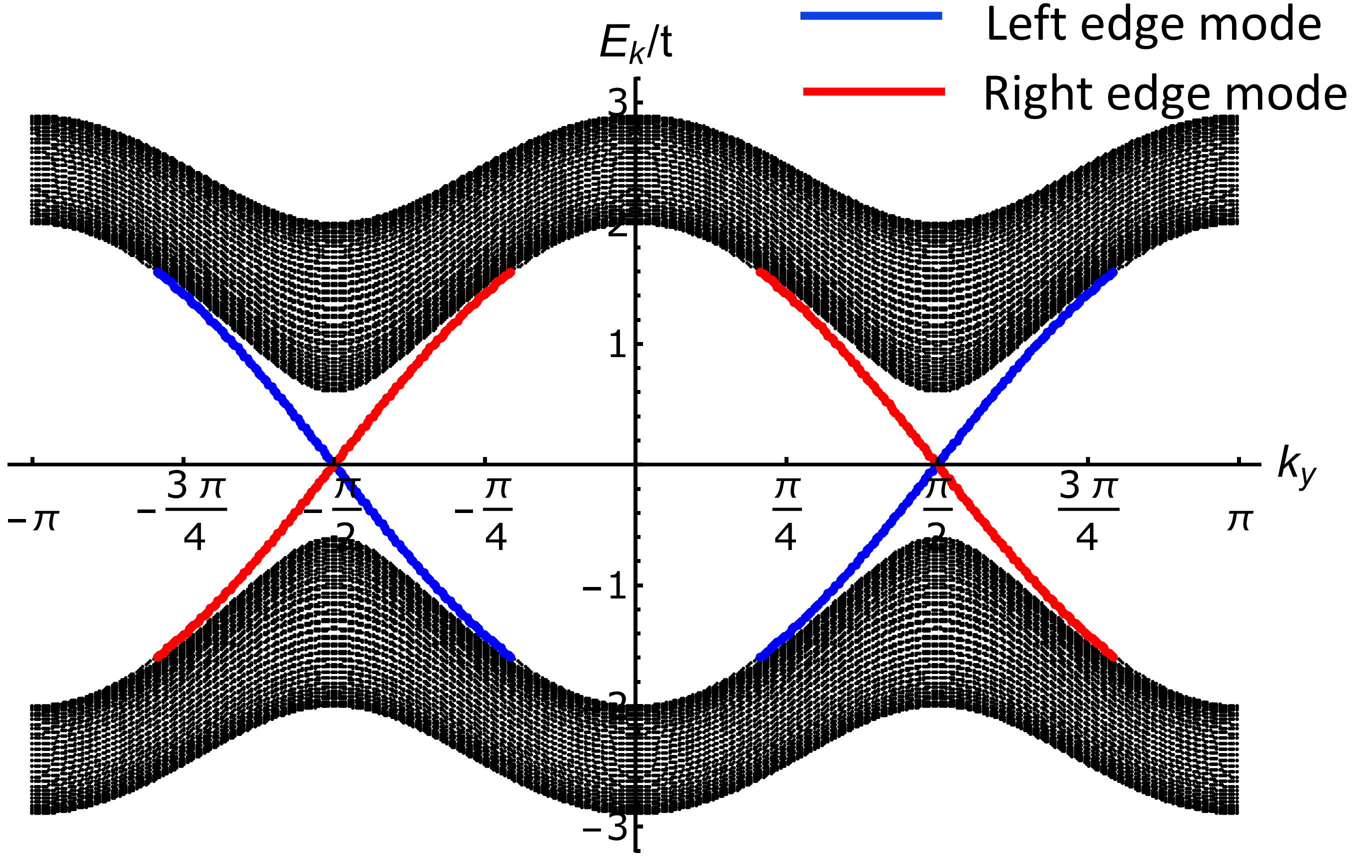
Wu, Ho and YML, arXiv ***1703.04776*** *(2017)*

Figure from Qi et al (2009)

Example 2

π-flux on square lattice with spin-orbit coupling





Wu, Ho and YML, arXiv ***1703.04776*** *(2017)*

Energy spectrum with open boundary in *x*-direction and periodic in *y*.

Application 3: Majorana vortex lattice (*class D*)

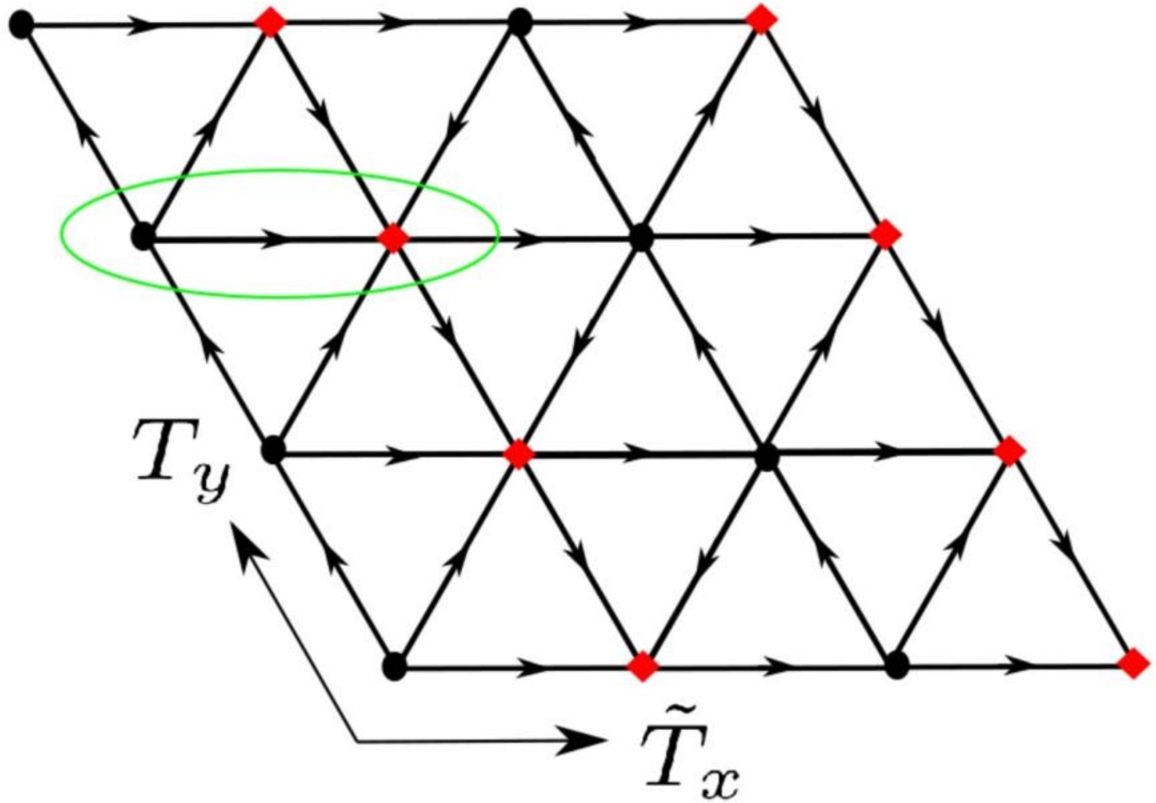
- **Input:** $\bar{\rho}_f \in \mathbb{Z}$ Majoranas **+** $\phi = \pi$ flux per u.c.
- **Output:** Topological index $\nu = \bar{\rho}_f \mod 2$ for any unique gapped ground state

- *With **odd Majoranas per unit cell** , a unique gapped g.s. must be **a chiral TSC with a half-integer chiral central charge**!*

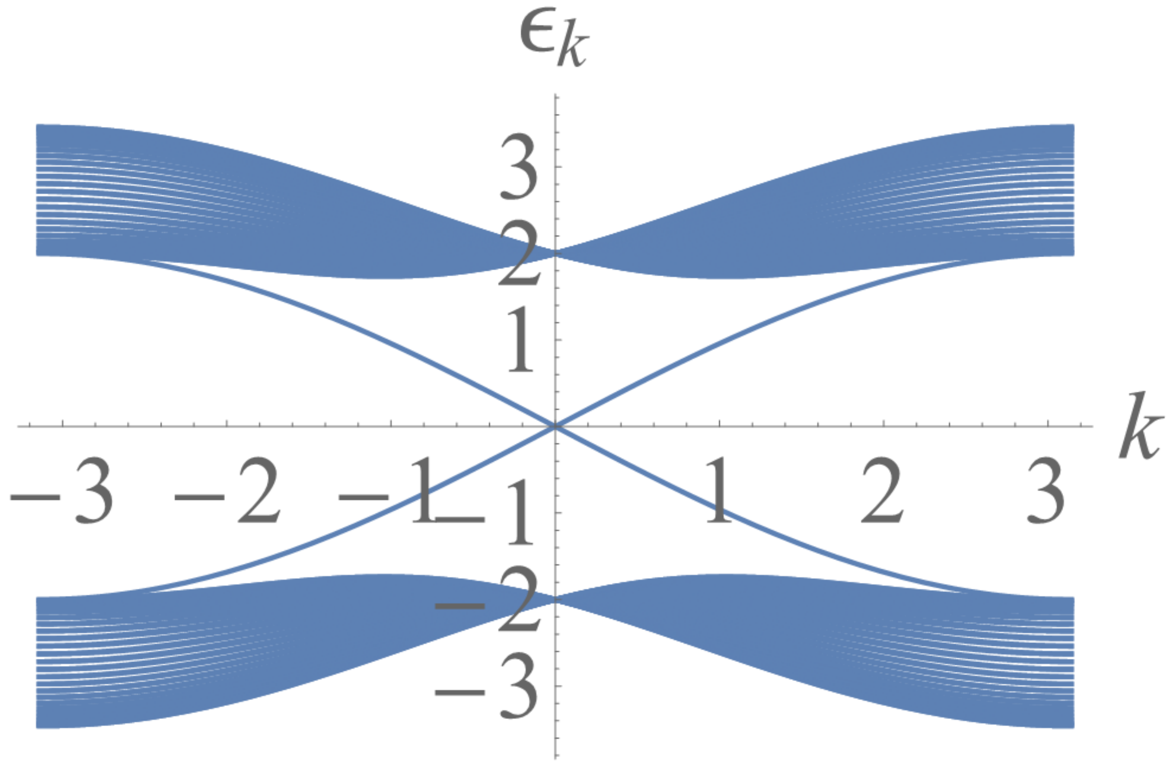
*YML, arXiv **1705.04691***

Figure from Qi et al (2009)

Example 3 Majorana vortex lattice of *p+ip* superconductor



Grosfeld, Stern (2006)



Chiral Majorana edge mode with *c_=1/2*

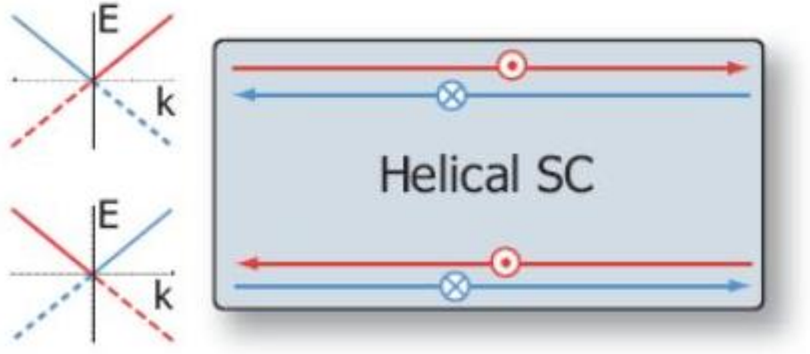
YML, arXiv ***1705.04691***

Application 4: Quantum spin liquids (*class DIII*)

- **Input:** $\bar{\rho}_f \in \mathbb{Z}$ Majorana Kramers pairs per u.c. **+** $\phi = \pi$ flux **+ *Time reversal symmetry***

- **Output:** Topological index $\nu = \bar{\rho}_f \mod 2$ for any unique gapped ground state

- *A unique gapped g.s.* ***must be helical TSC with odd Majorana pairs per unit cell !***

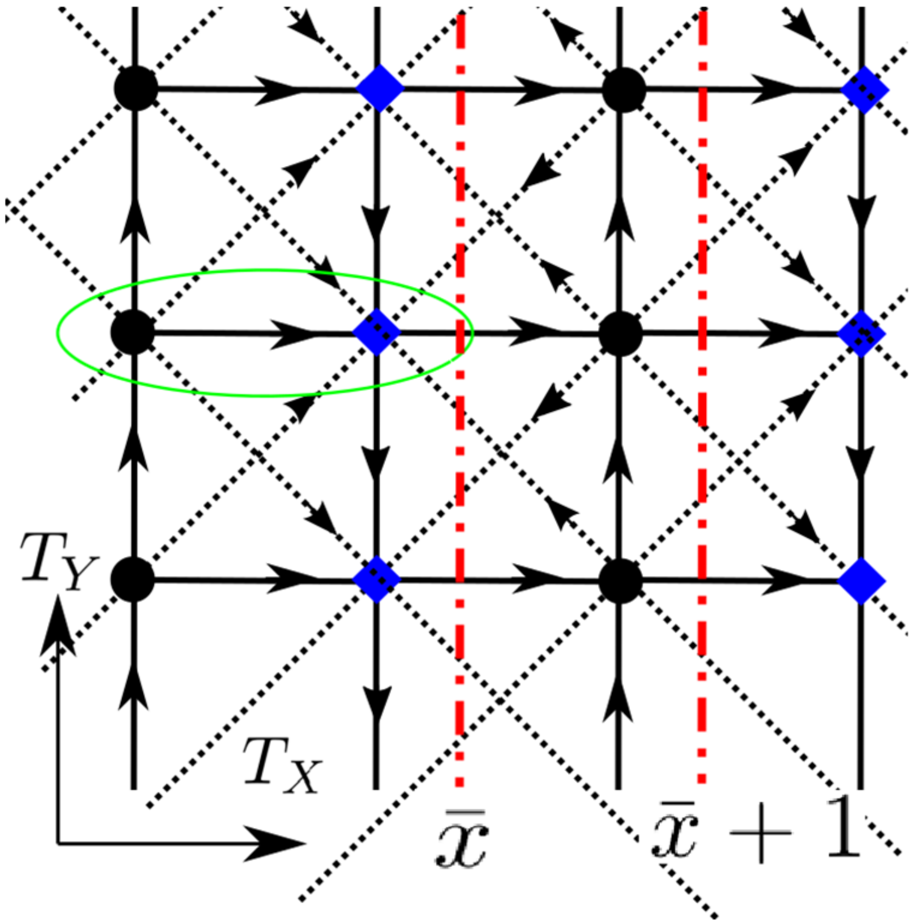


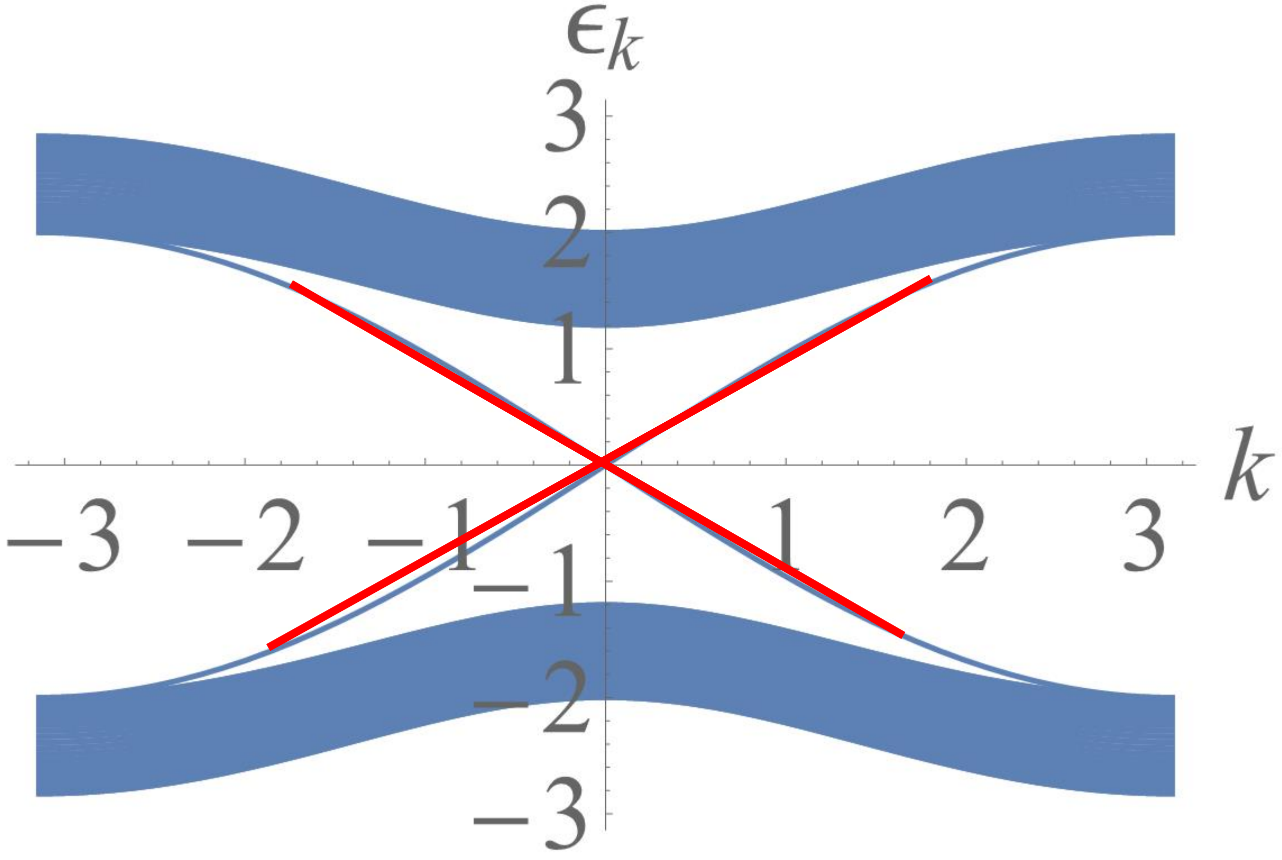
massless Majorana fermions

Figure from Qi et al (2009)

YML, arXiv ***1705.04691***

Example 4 Kitaev-type Z_2 spin liquids on square lattice





Helical Majorana edge modes

Toric code where TRS exchanges *e* and *m*

Nakai, Ryu, Furusaki (2012)

YML, arXiv ***1705.04691***

LSM theorem for bosonic SPT phases in *d=2*

IQHE of interacting bosons

QSHE of interacting bosons

Global symmetries and SPT classification			Microscopic input		Output of LSM theorem
Symmetry group G_s	Classification $\mathcal{H}^3(G_s, U(1))$	Topological invariants	Density/d.o.f. per unit cell	Flux per unit cell	Topological index
$U(1)$	$2\mathbb{Z}$	$\sigma_{xy} = \text{even}$	$\bar{\rho} = \frac{2a}{q}$	$\phi = 2\pi\frac{p}{q}$	$p \cdot \sigma_{xy} = 2a \mod q$ (BIQH states)
$U(1) \rtimes Z_2^{\mathcal{T}}$	$\mathbb{Z}_2$	$\nu_{\mathcal{T}} = 0, 1 \in \mathbb{Z}_2$	$\bar{\rho} \in \mathbb{Z}$ + an odd number of Kramers doublets	$\phi = \pi$	$\nu_{\mathcal{T}} = 1$ (BQSH state)
$Z_2 \times Z_2^{\mathcal{T}}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\nu, \nu_{\mathcal{T}} = 0, 1 \in \mathbb{Z}_2$	an odd number of Kramers doublets	$\phi = \pi$	$\nu_{\mathcal{T}} = 1$
$U(1)_A \times U(1)_B$	$(2\mathbb{Z})^2 \times \mathbb{Z}$	$\sigma_{xy}^A, \sigma_{xy}^B = \text{even}$ $\sigma_{xy}^{AB} = \sigma_{xy}^{BA} \in \mathbb{Z}$	$(\bar{\rho}_A, \bar{\rho}_B)$	(ϕ_A, ϕ_B)	$(\sigma_{xy}^A, \sigma_{xy}^B, \sigma_{xy}^{AB})$ satisfying $\boxed{9}$)
$U(1)_A \times (Z_q)_B$	$2\mathbb{Z} \times (\mathbb{Z}_q)^2$	$\sigma_{xy}^A = \text{even}$ $\nu^B, \nu^{AB} \in \mathbb{Z}_q$	$\bar{\rho}_A = \frac{a}{q}$	$\phi_B = 2\pi\frac{p}{q}$ $\phi_A = 0, \pi$	$p \cdot \nu^{AB} = a \mod q$

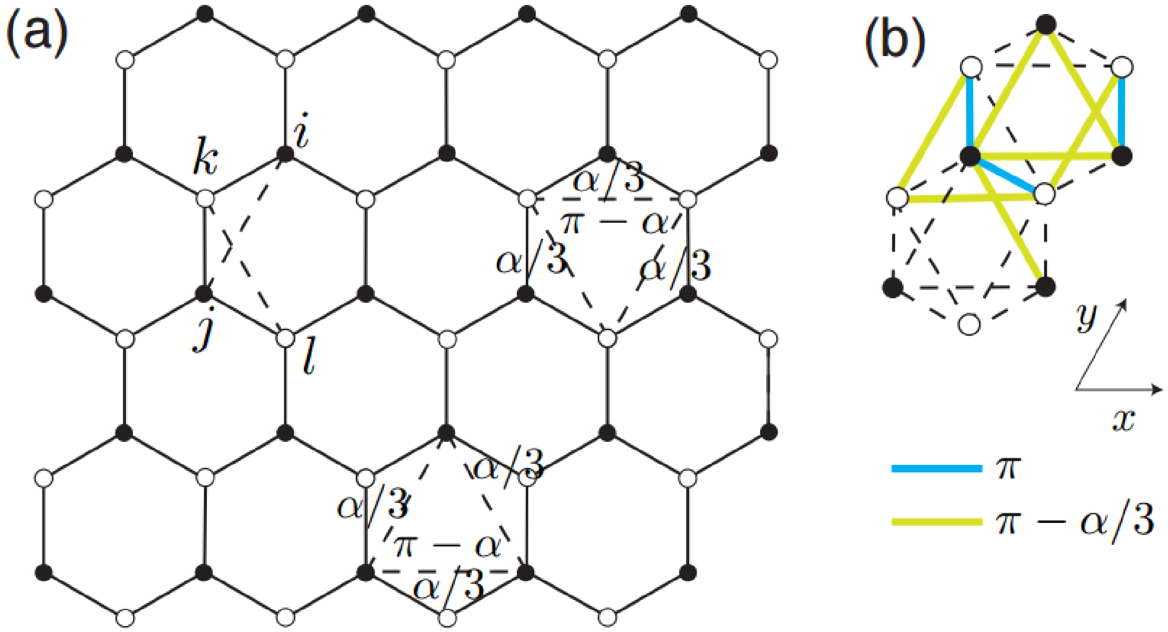
See also *Yang, Jiang, Vishwanath, Ran; arXiv* ***1705.05421***

"LSM theorem for SPT phases", YML, arXiv ***1705.04691***

Application 5: IQHE of two-component bosons

$$G_s = U(1)_A \times U(1)_B$$

$$\frac{1}{2\pi}\begin{pmatrix} \sigma_{xy}^{A} & \sigma_{xy}^{AB} \\ \sigma_{xy}^{BA} & \sigma_{xy}^{B} \end{pmatrix}\begin{pmatrix} \phi_A \\ \phi_B \end{pmatrix} = \begin{pmatrix} \bar{\rho}_A \\ \bar{\rho}_B \end{pmatrix} \mod 1$$



Yin-Chen He et al (2015)

Input: $\bar{\rho}_A = \bar{\rho}_B = \frac{1}{2}$

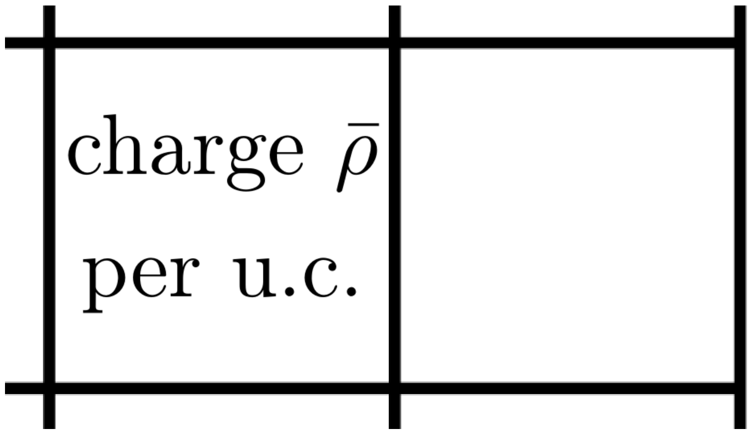
$\phi_A = \phi_B = \pi$

Output:

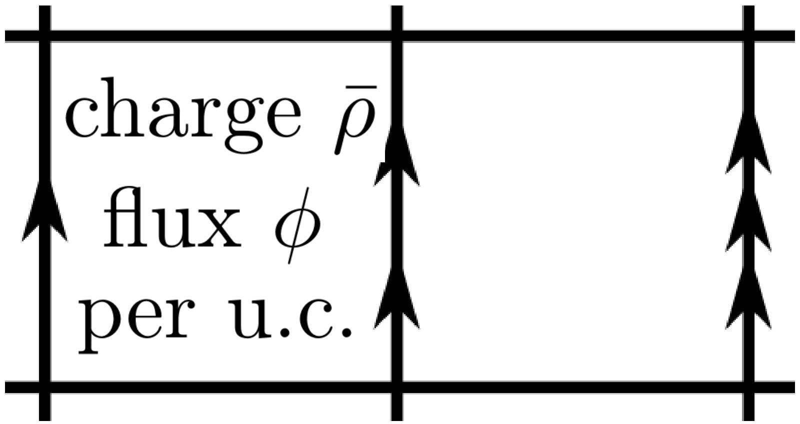
$$\Longrightarrow \sigma_{xy}^{AB} = 1 \mod 2$$

YML, arXiv ***1705.04691***

Physical intuition: charge-flux binding in SPT phases



$\longrightarrow T_x$



$\longrightarrow \tilde{T}_x = T_x \cdot e^{\mathrm{i}\phi\sum_{\mathbf{r}} y\hat{n}_{\mathbf{r}}}$

The "Old" LSM theorem:

Unique gapped g.s. → $\bar{\rho} \in \mathbb{Z}$

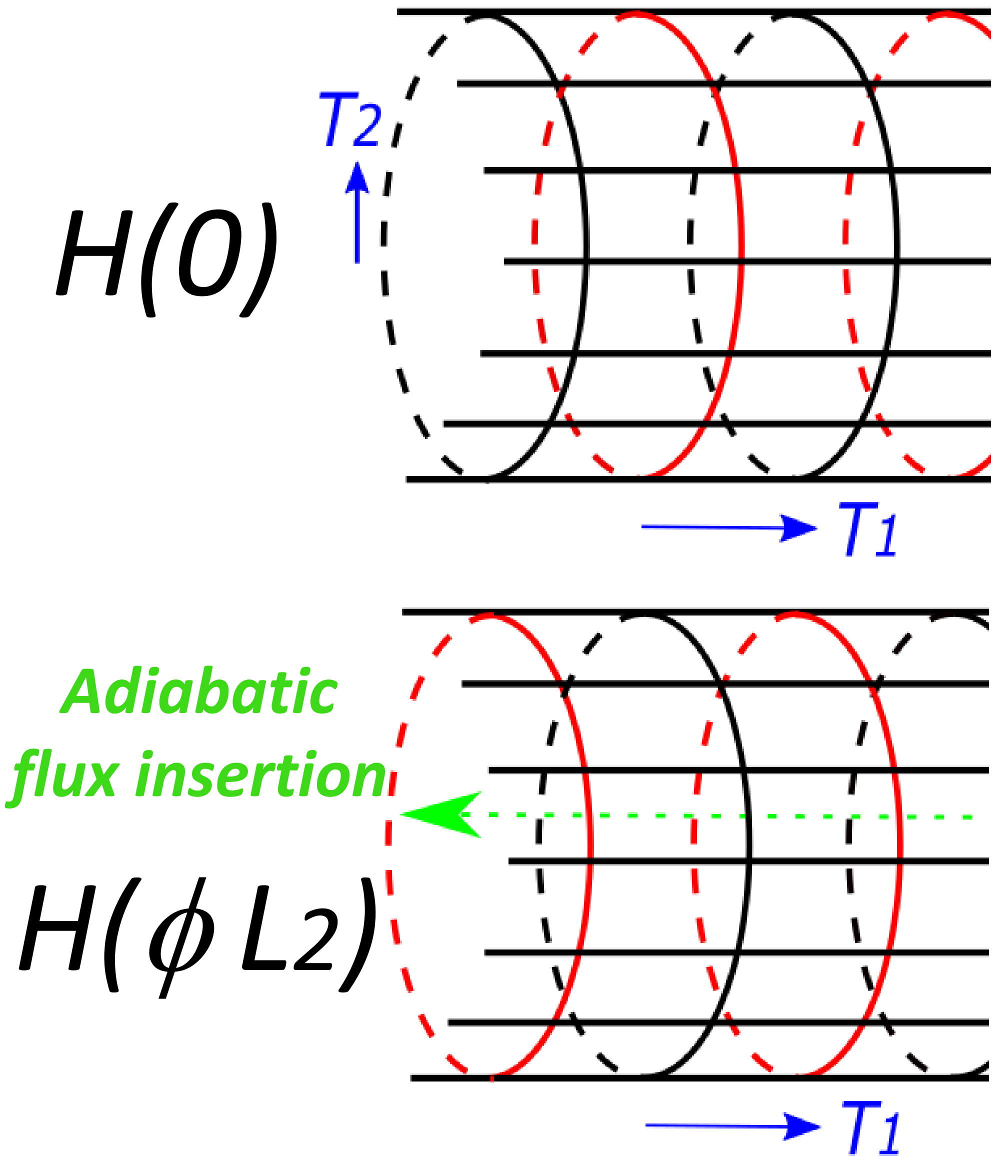
QHE → "background charge"

$\Delta\rho = -\sigma_{xy}\frac{\phi}{2\pi}$ bound to flux ϕ

Our "New" LSM theorem:

Unique gapped g.s. → $\bar{\rho} + \Delta\rho \in \mathbb{Z}$

Setup for the proof



Infinite cylinder (or torus)
periodic along *T2* direction

$$\phi = \pi, \quad L_2 = 1 \mod 2$$

$$\mathcal{F}_2(\Phi) = \mathcal{T} e^{-\mathrm{i} \int_0^T \hat{H}(\Phi(\tau)) \mathrm{d}\tau}$$

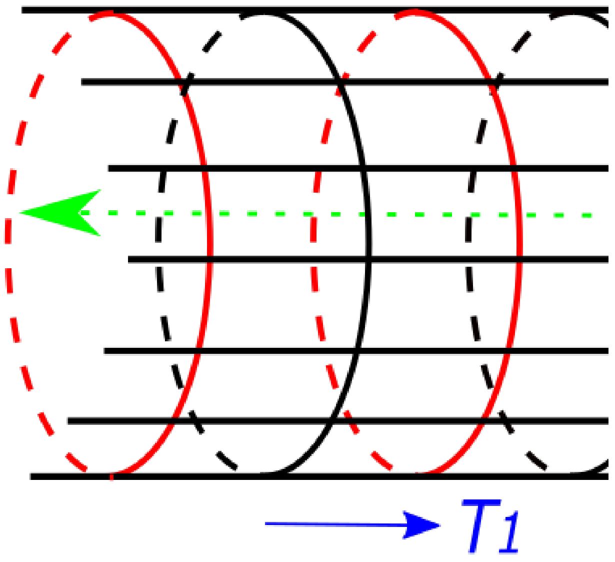
Adiabatic evolution of a unique gapped g.s.

$$T_1^{-1} \mathcal{F}_2(\phi L_2) |\Psi(0)\rangle = e^{\mathrm{i}\alpha} |\Psi(0)\rangle$$

Effective sym. on g.s.

$$|\Psi(\phi L_2)\rangle = e^{\mathrm{i}\alpha} T_1 |\Psi(0)\rangle$$

LSM for **IQHE** (*Class A*): polarization



Adiabatic flux insertion

Center of mass position of all charges

$$P_1 \equiv \langle e^{\frac{2\pi \mathrm{i}}{L_1} \sum_{\mathbf{x}} x_1 \hat{n}_{\mathbf{x}}} \rangle$$

"Order parameter" for an (interacting) insulator

(*King-Smith, Vanderbilt 1993, Resta, Sorella, 1999……*)

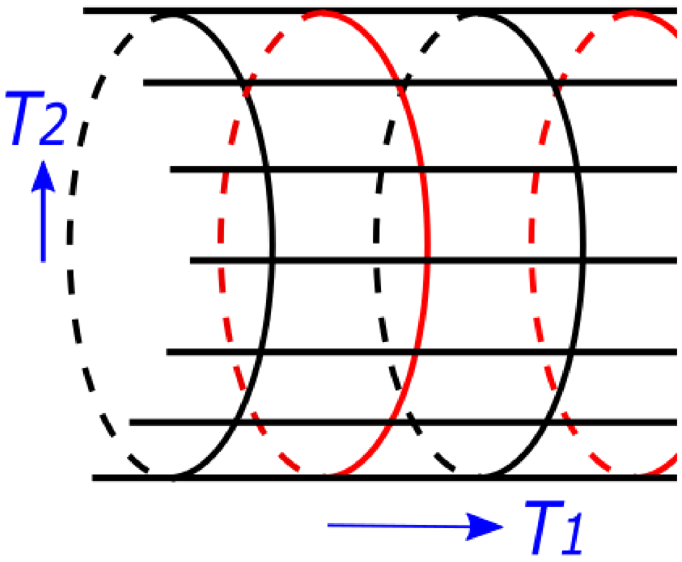
$$[T_1^{-1}\mathcal{F}_2(\phi L_2), P_1] = 0 \Longrightarrow$$

$$e^{\mathrm{i}\, L_2 (\sigma_{xy}\phi - 2\pi\bar{\rho})} = 1$$

$$T_1^{-1}\mathcal{F}_2(\phi L_2)|\Psi(0)\rangle = e^{\mathrm{i}\,\alpha}|\Psi(0)\rangle$$

YML, Ran, Oshikawa, arXiv ***1705.09298***

LSM for chiral TSC (*Class D*): fermion parity

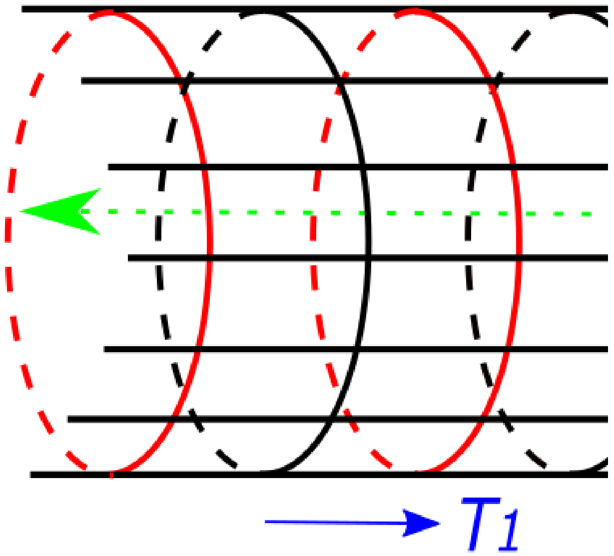


$$|\Psi(0)\rangle$$

$$\{\hat{T}_1, (-1)^{\hat{F}}\} = 0 \quad \text{if} \quad L_2 = \text{odd}$$

Translation *T1* of odd Majoranas is a SUSY

(*Hsieh, Halasz, Grover, 2016*)

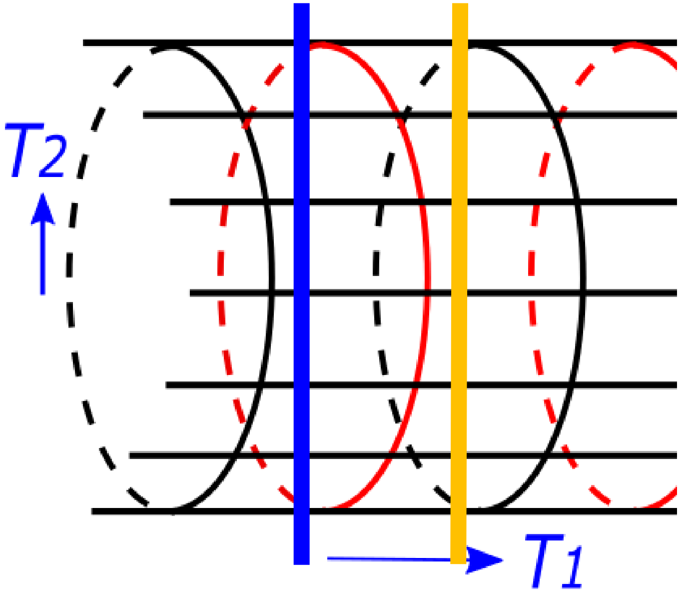


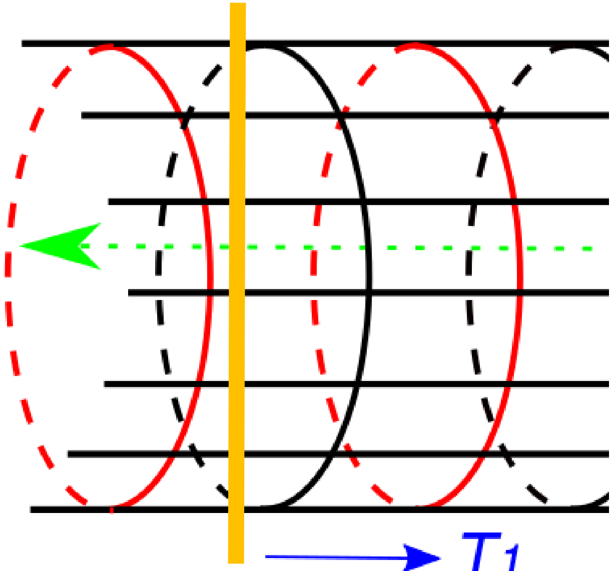
Twist boundary condition → switch fermion parity

➔Must be a chiral TSC of *p+ip* type; with an odd # of MZMs in each π-flux (*Read, Green 2000*)

$$|\Psi(\pi L_2)\rangle = e^{\mathrm{i}\,\alpha}\hat{T}_1|\Psi(0)\rangle$$

LSM for **QSHE/SPT:** entanglement spectrum

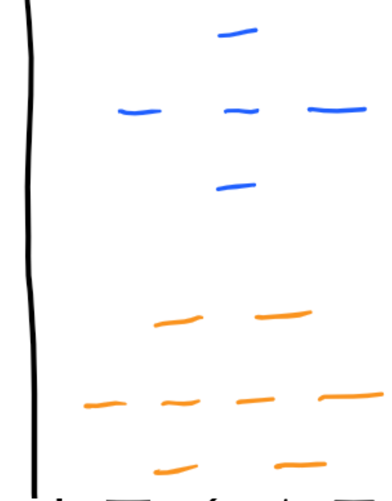




TRS → π-flux: $\phi = \pi, \;\; \bar{\rho}_f = \text{odd}, \; L_2 = \text{odd}$

$$|\Psi(0)\rangle = \sum_\alpha \lambda_{\bar{x},\alpha} |\alpha\rangle_{\bar{x},L} |\alpha\rangle_{\bar{x},R}$$

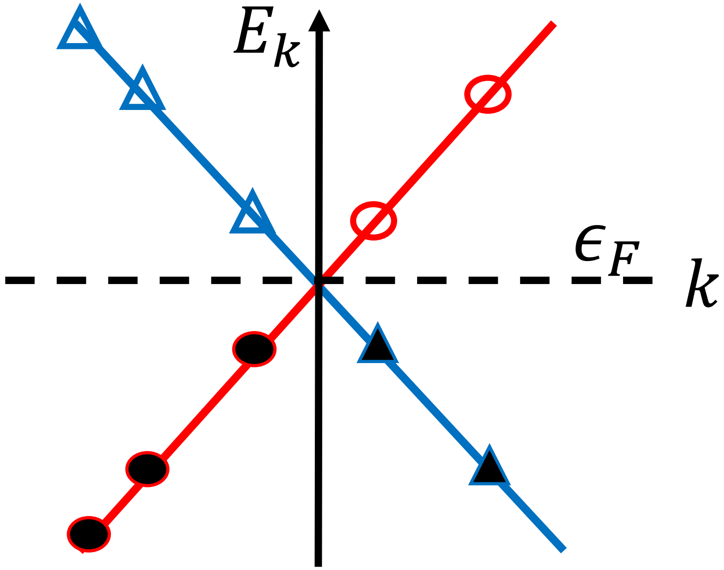
$\lambda_\alpha = e^{-\epsilon_\alpha}$



Kramers singlet (PBC)

vs.

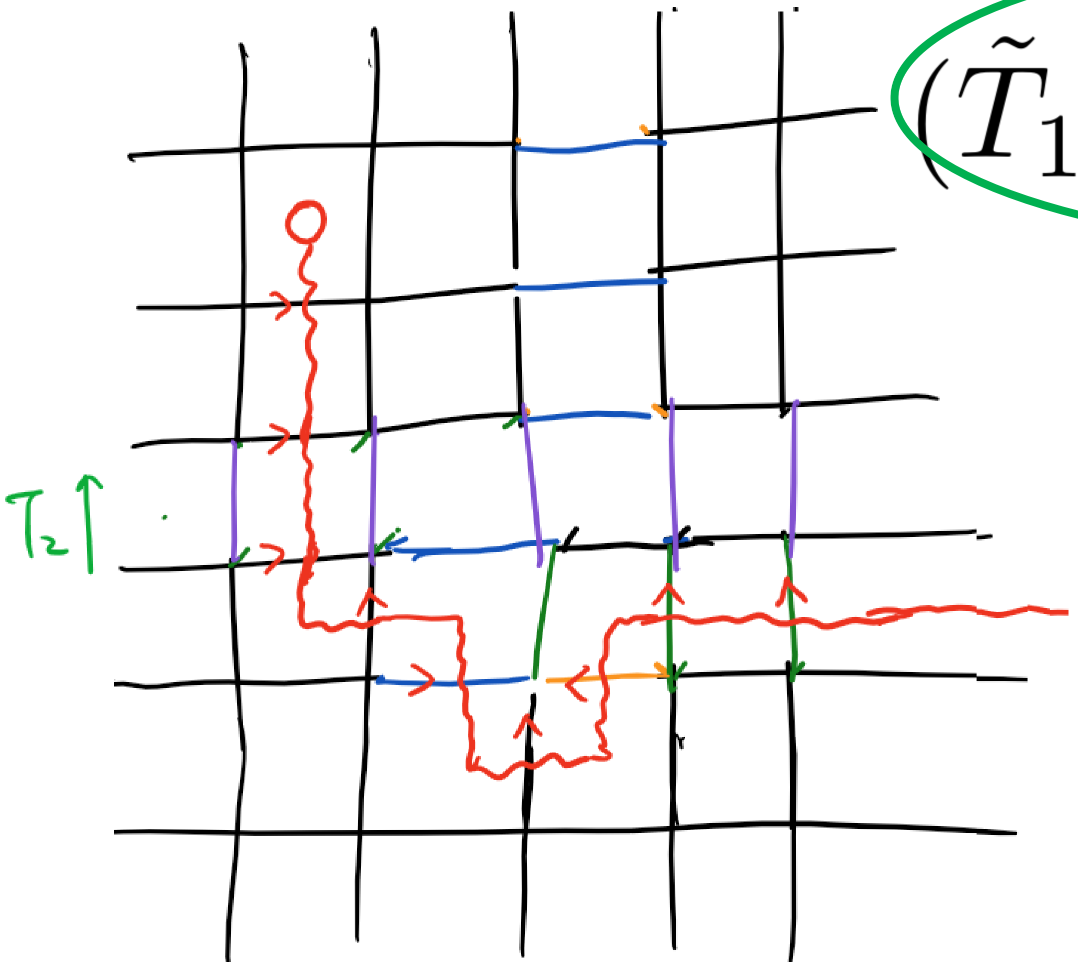
Kramers doublet (APBC)



$$|\Psi(\phi L_2)\rangle = e^{\imath\alpha} T_1 |\Psi(0)\rangle$$

Wu, Ho and YML, arXiv ***1703.04776*** *(2017)*

Generalization to SETs: FQHEs with U(1) sym.



$$(\tilde{T}_1)^{-1}(\tilde{T}_2)^{-1}\tilde{T}_1\tilde{T}_2 e^{-\mathrm{i}\phi\hat{N}} = 1$$

$$e^{2\pi\mathrm{i}\bar{\rho}} \cdot e^{-\mathrm{i}\phi\sigma_{xy}} = e^{\mathrm{i}\theta_{F,a}}$$

$$\bar{\rho} - \frac{\phi}{2\pi}\sigma_{xy} = \frac{\theta_{F,a}}{2\pi} \mod 1$$

"background anyon" ***a*** per unit cell

$$e^{\mathrm{i}\theta_{F,F}} = e^{2\pi\mathrm{i}\sigma_{xy}}$$

*YML, Ran, Oshikawa, arXiv **1705.09298***

Concluding Remarks

- "Old" LSM theorem (w/ translation sym.): ***Impossible to have a unique gapped (SRE) ground state*** (*No-go theorem*)

- "New" LSM theorem (w/ **magnetic** translation sym.):
  If a unique gapped g.s. (SRE) → must be a SPT (invertible) phase!
- Applies to interacting bosons/fermions in $d=2$ w/ various global symmetries

- *Realistic models and numerics?*
- *Generalization to $d>2$?*
- *Generalization to space group sym.?*
- *Generalization to symmetry-enriched topological (SET) orders?*

Collaborators and References

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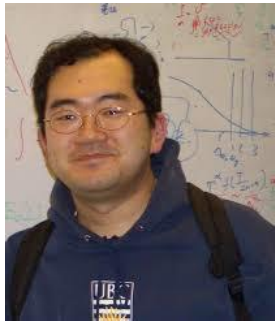
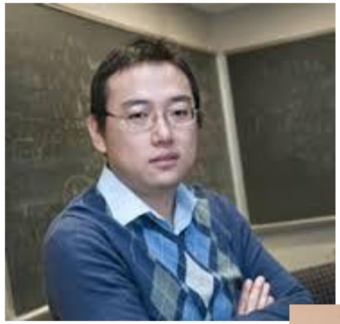


❑ *"Filling-enforced constraint on the quantized Hall conductivity on a periodic lattice"*, YML, Ran, Oshikawa, arxiv ***1705.09298***

❑ *"Symmetry-enforced QSHE in π-flux models",* Wu, Ho, YML, *arXiv* ***1703.04776***

❑ *"LSM theorems for SPT phases",* YML, arxiv ***1705.04691***

❑ Fuyan Lu, YML, in preparation (2017)